Large-Scale Knowledge Processing

Optimization Techniques (3) Supplementary Materials
Practice Brief Solutions

p.9 Practice

(a) From the standard form, we construct the following artificial problem.

$$\begin{array}{lll} \text{minimize} & w = & x_4 \\ \text{subject to} & & -x_1 - x_2 - x_3 + x_4 = 1 \\ & & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

We solve this problem by the simplex method.

		x_1	x_2	x_3
w	1	1	1	1
x_4	1	1	1	1

Since the optimal value of the artificial problem satisfies $w^* > 0$, we conclude that the original problem is infeasible.

(b) The standard form is given as follows.

$$\begin{array}{lll} \text{minimize} & z = & -x_1 - 2x_2 \\ \text{subject to} & & -x_1 - x_2 + x_3 = 1 \\ & & x_1, x_2, x_3 \geq 0 \end{array}$$

From the standard form, we construct the following artificial problem.

$$\begin{array}{lll} \text{minimize} & w = & x_4 \\ \text{subject to} & & -x_1 - x_2 + x_3 + x_4 = 1 \\ & & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

We solve this problem by the simplex method.

		x_1	x_2	x_3
w	1	1	1	-1
x_4	1	1	1	-1

Pivot at $(1, 3)$

		x_1	x_2	x_4
w	0	0	0	1
x_3	1	1	1	-1

The artificial problem has optimal value $w^* = 0$, and the set of basic variables does not contain the artificial variable x_4. Hence, $(x_1, x_2, x_3) = (0, 0, 1)$ is a feasible basic solution of the problem in standard form.

We now solve the problem in standard form by the simplex method.

		x_1	x_2
z	0	-1	-2
x_3	1	1	1

As x_2 increases, x_3 also increases, and thus the problem is unbounded (that is, the objective value can be decreased without bound).

(c) The standard form is given as follows.

$$\begin{array}{ll} \text{minimize} & z = -x_1 - 2x_2 \\ \text{subject to} & x_1 + x_2 - x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

From the standard form, we construct the following artificial problem.

$$\begin{array}{ll} \text{minimize} & w = x_4 \\ \text{subject to} & x_1 + x_2 - x_3 + x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

We solve this problem by the simplex method.

		x_1	x_2	x_3
w	1	-1	-1	1
x_4	1	-1	-1	1

Pivot at $(1, 1)$

		x_4	x_2	x_3
w	0	1	0	0
x_1	1	-1	-1	1

The artificial problem has optimal value $w^* = 0$, and the set of basic variables does not contain the artificial variable x_4. Hence, $(x_1, x_2, x_3) = (1, 0, 0)$ is a feasible basic solution of the problem in standard form.

We now solve the problem in standard form by the simplex method.

		x_2	x_3
z	-1	-1	-1
x_1	1	-1	1

Pivot at $(1, 1)$

		x_1	x_3
z	-2	1	-2
x_2	1	-1	1

As x_3 increases, x_2 also increases, and thus the problem is unbounded (that is, the objective value can be decreased without bound).

p.10 Practice

(a) From the standard form, we construct the following artificial problem.

$$\begin{array}{ll} \text{minimize} & w = x_3 + x_4 \\ \text{subject to} & -x_1 - x_2 + x_3 \quad = 2 \\ & -2x_1 - x_2 \quad + x_4 = 8 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

We solve this problem by the simplex method.

		x_1	x_2
w	10	3	2
x_3	2	1	1
x_4	8	2	1

Since the optimal value of the artificial problem satisfies $w^* > 0$, the original problem is infeasible.

p.10 Practice

(b) The standard form is given as follows.

$$\begin{array}{ll} \text{minimize} & z = 3x_1 + x_2 \\ \text{subject to} & x_1 + x_2 + x_3 = 4 \\ & x_1 + 2x_2 = 5 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

From the standard form, we construct the following artificial problem.

$$\begin{array}{ll} \text{minimize} & w = x_4 + x_5 \\ \text{subject to} & x_1 + x_2 + x_3 + x_4 = 4 \\ & x_1 + 2x_2 + x_5 = 5 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}$$

We solve this problem by the simplex method.

		x_1	x_2	x_3
w	9	-2	-3	-1
x_4	4	-1	-1	-1
x_5	5	-1	-2	0

Pivot at $(2, 2)$

		x_1	x_5	x_3
w	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	-1
x_4	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1
x_2	$\frac{5}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0

Pivot at $(1, 3)$

		x_1	x_5	x_4
w	0	0	1	1
x_3	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1
x_2	$\frac{5}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0

The artificial problem has optimal value $w^* = 0$, and the set of basic variables does not contain the artificial variables x_4 and x_5. Hence, $(x_1, x_2, x_3) = (0, \frac{5}{2}, \frac{3}{2})$ is a feasible basic solution of the problem in standard form.

We now solve the problem in standard form by the simplex method.

		x_1
z	$\frac{5}{2}$	$\frac{5}{2}$
x_3	$\frac{3}{2}$	$-\frac{1}{2}$
x_2	$\frac{5}{2}$	$-\frac{1}{2}$

Therefore, the problem in standard form attains the optimal value $\frac{5}{2}$ at $(x_1, x_2) = (0, \frac{5}{2})$.
The optimal value of the original problem is $-\frac{5}{2}$.

p.11 Practice

(a) The standard form is given as follows.

$$\begin{array}{ll} \text{minimize} & z = -x_1 - 4x_2 \\ \text{subject to} & 2x_1 + x_2 + x_3 = 8 \\ & x_1 + 2x_2 + x_4 = 10 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

From the standard form, we construct the following artificial problem.

$$\begin{array}{lrl} \text{minimize} & w = x_5 + x_6 & \\ \text{subject to} & 2x_1 + \ \ x_2 + x_3 \qquad + x_5 \qquad & = 8 \\ & x_1 + 2x_2 \qquad + x_4 \qquad + x_6 & = 10 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 & \end{array}$$

We solve this problem by the simplex method.

		x_1	x_2	x_3	x_4
w	18	-3	-3	-1	-1
x_5	8	-2	-1	-1	0
x_6	10	-1	-2	0	-1

Pivot at $(1, 1)$

		x_5	x_2	x_3	x_4
w	6	$\frac{3}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	-1
x_1	4	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0
x_6	6	$\frac{1}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	-1

Pivot at $(2, 2)$

		x_5	x_6	x_3	x_4
w	0	1	1	0	0
x_1	2	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$
x_2	4	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$

The artificial problem has optimal value $w^* = 0$, and the set of basic variables does not contain the artificial variables x_5 and x_6. Hence, $(x_1, x_2, x_3, x_4) = (2, 4, 0, 0)$ is a feasible basic solution of the problem in standard form.

We now solve the problem in standard form by the simplex method.

		x_3	x_4
z	-18	$-\frac{2}{3}$	$\frac{7}{3}$
x_1	2	$-\frac{2}{3}$	$\frac{1}{3}$
x_2	4	$\frac{1}{3}$	$-\frac{2}{3}$

Pivot at $(1, 1)$

		x_1	x_4
z	-20	1	2
x_3	3	$-\frac{3}{2}$	$\frac{1}{2}$
x_2	5	$-\frac{1}{2}$	$-\frac{1}{2}$

Therefore, the problem in standard form attains the optimal value -20 at $(x_1, x_2, x_3, x_4) = (0, 5, 3, 0)$. The original problem attains the optimal value 20 at $(x_1, x_2) = (0, 5)$.

(b) The standard form is given as follows.

$$\begin{array}{lrl} \text{minimize} & z = -x_1 - 4x_2 & \\ \text{subject to} & 2x_1 + \ \ x_2 - x_3 \qquad & = 8 \\ & x_1 + 2x_2 \qquad - x_4 & = 10 \\ & x_1, x_2, x_3, x_4 \geq 0 & \end{array}$$

From the standard form, we construct the following artificial problem.

$$
\begin{array}{lrl}
\text{minimize} & w = x_5 + x_6 & \\
\text{subject to} & 2x_1 + x_2 - x_3 \qquad + x_5 \qquad & = 8 \\
& x_1 + 2x_2 \qquad - x_4 \qquad + x_6 & = 10 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 &
\end{array}
$$

We solve this problem by the simplex method.

		x_1	x_2	x_3	x_4
w	18	-3	-3	1	1
x_5	8	-2	-1	1	0
x_6	10	-1	-2	0	1

Pivot at $(1,1)$

		x_5	x_2	x_3	x_4
w	6	$\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	1
x_1	4	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0
x_6	6	$\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	1

Pivot at $(2,2)$

		x_5	x_6	x_3	x_4
w	0	1	1	0	0
x_1	2	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$
x_2	4	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$

The artificial problem has optimal value $w^* = 0$, and the set of basic variables does not contain the artificial variables x_5 and x_6. Hence, $(x_1, x_2, x_3, x_4) = (2, 4, 0, 0)$ is a feasible basic solution of the problem in standard form.

We now solve the problem in standard form by the simplex method.

		x_3	x_4
z	-18	$\frac{2}{3}$	$-\frac{7}{3}$
x_1	2	$\frac{2}{3}$	$-\frac{1}{3}$
x_2	4	$-\frac{1}{3}$	$\frac{2}{3}$

Pivot at $(1,2)$

		x_3	x_1
z	-32	-4	7
x_4	6	2	-3
x_2	8	1	-2

As x_3 increases, both x_2 and x_4 increases as well, and thus the problem is unbounded (that is, the objective value can be decreased without bound).