Large-Scale Knowledge Processing

Optimization Techniques (3)
Practice Brief Solutions

p.7 Practice

(a) Let x_3 and x_4 be the basic variables.

		x_1	x_2
z	0	-4	-6
x_3	4	-2	-2
x_4	9	-3	-6

Supplementary explanation:

Since the reduced cost coefficients of both x_1 and x_2 are negative, we choose x_2 (the second column), which has the larger absolute value, as the pivot-in variable. Keeping $x_1 = 0$ fixed, increase x_2 from 0. If x_2 exceeds $\frac{3}{2}$, the constraint $x_4 \geq 0$ is violated. Therefore, we choose x_4 (the second row) as the pivot-out variable.

Pivot at $(2, 2)$

Swap the variable in the second row with the variable in the second column.

		x_1	x_4
z	-9	-1	1
x_3	1	-1	$\frac{1}{3}$
x_2	$\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{1}{6}$

Only the reduced cost coefficient of x_1 (the first column) is negative, so we choose it as the pivot-in variable. Keeping $x_4 = 0$ fixed, increase x_1 from 0. If x_1 exceeds 1, the constraint $x_3 \geq 0$ is violated. Therefore, we choose x_3 (the first row) as the pivot-out variable.

Pivot at $(1, 1)$

Swap the variable in the first row with the variable in the first column.

		x_3	x_4
z	-10	1	$\frac{2}{3}$
x_1	1	-1	$\frac{1}{3}$
x_2	1	$\frac{1}{2}$	$-\frac{1}{3}$

Since the reduced cost coefficients of all nonbasic variables are nonnegative, we conclude that an optimal solution has been obtained.

Therefore, the optimal solution is $x = (1, 1, 0, 0)$, and the optimal value is -10.

※ The operations in the first pivot at $(2, 2)$ are the same as those in "Optimization Techniques (2)" slide p.24. Today's approach is recommended, since it avoids repeatedly writing variable names, unlike the previous method.

(b) Let x_3 and x_4 be the basic variables.

		x_1	x_2
z	0	-4	-5
x_3	4	-2	-2
x_4	8	-3	-6

Pivot at $(2, 2)$

		x_1	x_4
z	$-\frac{20}{3}$	$-\frac{3}{2}$	$\frac{5}{6}$
x_3	$\frac{4}{3}$	-1	$\frac{1}{3}$
x_2	$\frac{4}{3}$	$-\frac{1}{2}$	$-\frac{1}{6}$

Pivot at $(1, 1)$

		x_3	x_4
z	$-\frac{26}{3}$	$\frac{3}{2}$	$\frac{1}{3}$
x_1	$\frac{4}{3}$	-1	$\frac{1}{3}$
x_2	$\frac{2}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$

Therefore, the optimal solution is $x = (\frac{4}{3}, \frac{2}{3}, 0, 0)$, and the optimal value is $-\frac{26}{3}$.

(c) Let x_3, x_4, and x_5 be the basic variables.

		x_1	x_2
z	0	-4	-5
x_3	4	-2	-2
x_4	8	-3	-6
x_5	4	-1	-4

Pivot at $(3, 2)$

		x_1	x_5
z	-5	$-\frac{11}{4}$	$\frac{5}{4}$
x_3	2	$-\frac{3}{2}$	$\frac{1}{2}$
x_4	2	$-\frac{3}{2}$	$\frac{3}{2}$
x_2	1	$-\frac{1}{4}$	$-\frac{1}{4}$

Supplementary explanation: The pivot-in variable is x_1. Keeping $x_5 = 0$ fixed, increase x_1 from 0. If x_1 exceeds $\frac{4}{3}$, the constraints $x_3 \geq 0$ and $x_4 \geq 0$ can no longer be satisfied simultaneously. Either x_3 or x_4 may be chosen as the pivot-out variable; here we choose x_3, which has the smallest index.

Pivot at $(1, 1)$

		x_3	x_5
z	$-\frac{26}{3}$	$\frac{11}{6}$	$\frac{1}{3}$
x_1	$\frac{4}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$
x_4	0	1	1
x_2	$\frac{2}{3}$	$\frac{1}{6}$	$-\frac{1}{3}$

Therefore, the optimal solution is $x = (\frac{4}{3}, \frac{2}{3}, 0, 0, 0)$, and the optimal value is $-\frac{26}{3}$.

(d) Let x_1, x_2, and x_3 denote the production amounts of each product.

$$
\begin{aligned}
\text{maximize} \quad & z = 3x_1 + 5x_2 + 4x_3 \\
\text{subject to} \quad & 4x_1 + 2x_2 + x_3 \leq 6 \\
& x_1 + 2x_2 + 4x_3 \leq 7 \\
& 5x_1 + 2x_2 + 3x_3 \leq 9 \\
& 3x_1 + 3x_2 + 2x_3 \leq 8 \\
& x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0
\end{aligned}
$$

Transforming this problem into its standard form, we obtain the following.

$$
\begin{aligned}
\text{minimize} \quad & z = -3x_1 - 5x_2 - 4x_3 \\
\text{subject to} \quad & 4x_1 + 2x_2 + x_3 + x_4 = 6 \\
& x_1 + 2x_2 + 4x_3 + x_5 = 7 \\
& 5x_1 + 2x_2 + 3x_3 + x_6 = 9 \\
& 3x_1 + 3x_2 + 2x_3 + x_7 = 8
\end{aligned}
$$

We solve this problem using the simplex method.
Let x_4, x_5, x_6, and x_7 be the basic variables.

		x_1	x_2	x_3
z	0	-3	-5	-4
x_4	6	-4	-2	-1
x_5	7	-1	-2	-4
x_6	9	-5	-2	-3
x_7	8	-3	-3	-2

Pivot at $(4, 2)$

		x_1	x_7	x_3
z	$-\frac{40}{3}$	2	$\frac{5}{3}$	$-\frac{2}{3}$
x_4	$\frac{2}{3}$	-2	$\frac{2}{3}$	$\frac{1}{3}$
x_5	$\frac{5}{3}$	1	$\frac{2}{3}$	$-\frac{8}{3}$
x_6	$\frac{11}{3}$	-3	$\frac{2}{3}$	$-\frac{5}{3}$
x_2	$\frac{8}{3}$	-1	$-\frac{1}{3}$	$-\frac{2}{3}$

Pivot at $(2, 3)$

		x_1	x_7	x_5
z	$-\frac{55}{4}$	$\frac{7}{4}$	$\frac{3}{2}$	$\frac{1}{4}$
x_4	$\frac{7}{8}$	$-\frac{15}{8}$	$\frac{3}{4}$	$-\frac{1}{8}$
x_3	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{1}{4}$	$-\frac{3}{8}$
x_6	$\frac{21}{8}$	$-\frac{29}{8}$	$\frac{1}{4}$	$\frac{5}{8}$
x_2	$\frac{9}{4}$	$-\frac{5}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$

Therefore, the optimal solution to the original problem is $x = (0, \frac{9}{4}, \frac{5}{8})$, and the optimal value is $\frac{55}{4}$.

※ We present the optimal solution for the original problem, rather than for the problem in standard form. Note that, when transforming to standard form, the sign of the objective function is reversed; therefore, the sign of the optimal value must also be reversed accordingly.

p.15, p.17, p.21 Practice

Refer to the subsequent pages of the slide.