Large-Scale Knowledge Processing #3
Optimization Techniques (Part 1)

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Mathematical programming and optimization
Mathematical programming model

- Find the optimal solution
- What is the target for the optimization?
- Objective of the optimization?

Production planning problem

- Maximum profit ?
  We produce two products 1 and 2
  from two materials A and B
  - Material usage per unit of product and the amount of available materials

	Product 1 (10^3 kg)	Product 2 (10^3 kg)	Available materials (10^3 kg)
Material A	2	2	4
Material B	3	6	8

  - Profit per unit of product
    - Product 1: 4
    - Product 2: 5

Formulation of production planning

- Represent as mathematical models
- Let x_1, x_2 be the volumes of products 1, 2
- Objective： Maximize the profit
  - Maximize $4x_1 + 5x_2$
- Condition： Available materials
  - Material A： $2x_1 + 2x_2 \leqq 4$
  - Material B： $3x_1 + 6x_2 \leqq 8$
- Condition： Volumes of the products should be non-negative
  - Product 1： $x_1 \geqq 0$
  - Product 2： $x_2 \geqq 0$

Formulation of production planning

- Let x_1, x_2 be the volumes of products 1, 2
  ← decision variables, **variables**
- maximize $4x_1 + 5x_2$ ← **Objective function**

  subject to $2x_1 + 2x_2 \leqq 4$ ← **Constraints**

  $3x_1 + 6x_2 \leqq 8$

  $x_1 \geqq 0,\ x_2 \geqq 0$

- **Linear programming problem**
  - Objective function is a **linear** function
    - maximize or minimize the objective function
  - All constraints are **linear** inequalities or equalities

Production planning problem

- If the volumes of the products are limited to **integers**...
- maximize $4x_1 + 5x_2$

  subject to $2x_1 + 2x_2 \leqq 4$

  $3x_1 + 6x_2 \leqq 8$

  $x_1 \geqq 0,\ x_2 \geqq 0,\ x_1, x_2$ are integers

- **Integer programming problem**
  - Variables are limited to be integers
  - 0–1 integer programming problem

    … Variables are 0 or 1
  - Mixed integer programming problem

    … Some variables are integers/Others are real numbers

practice : Formulation of product planning

- Maximum profit ?
  We produce three products 1, 2 and 3 from four materials A, B, C and D
  - Material usage per unit of product and the amount of available materials

	Product 1 (10^3 kg)	Product 2 (10^3 kg)	Product 3 (10^3 kg)	Available Material (10^3 kg)
Material A	4	2	1	6
Material B	1	2	4	7
Material C	5	2	3	9
Material D	3	3	2	8

  - Profit per unit of product
    - Product 1 : 3, Product 2 : 5, Product 3 : 4

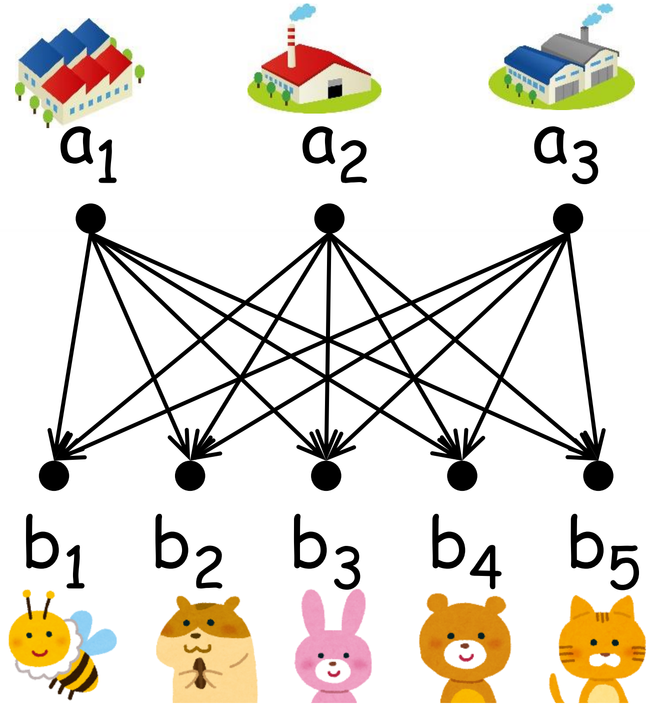
practice : Formulation of product planning

- Maximum profit ?

  We produce n kinds of products $P_1, P_2, \ldots, P_n$ from m kinds of materials $S_1, S_2, \ldots, S_m$

  - c_{ij} : Necessary amount of material S_j for producing a unit of product P_i
  - b_j : Available amount of material S_j
  - a_i : Profit per unit of product P_i

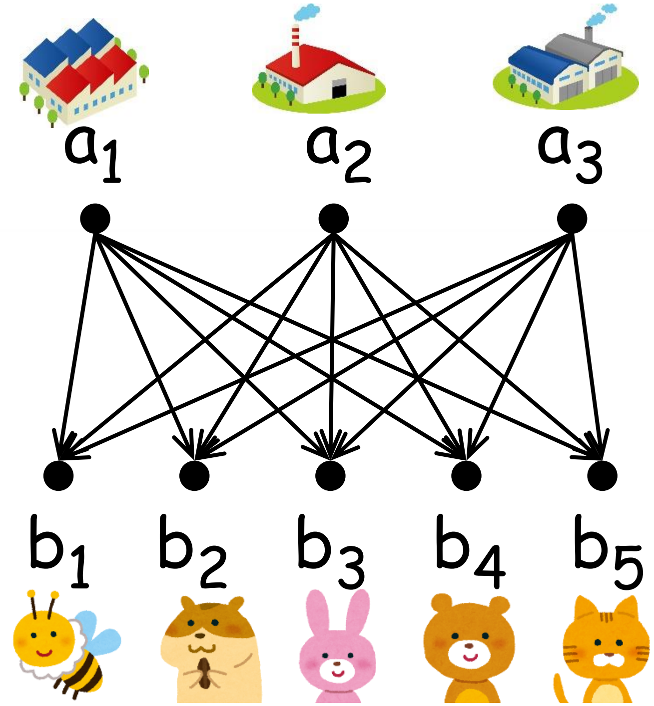
Transportation problem



- Transport products from factories 1, 2, 3 to consumers 1, 2, 3, 4, 5
- Minimize transportation costs

- a_i : The amount of products factory *i* can ship in a month
- b_j : Demand by consumer *j* in a month
- c_{ij} : Transportation cost (per unit of product) from factory *i* to consumer *j*

Transportation problem



- Let x_{ij} be the amount of the products shipped from factory i to consumer j

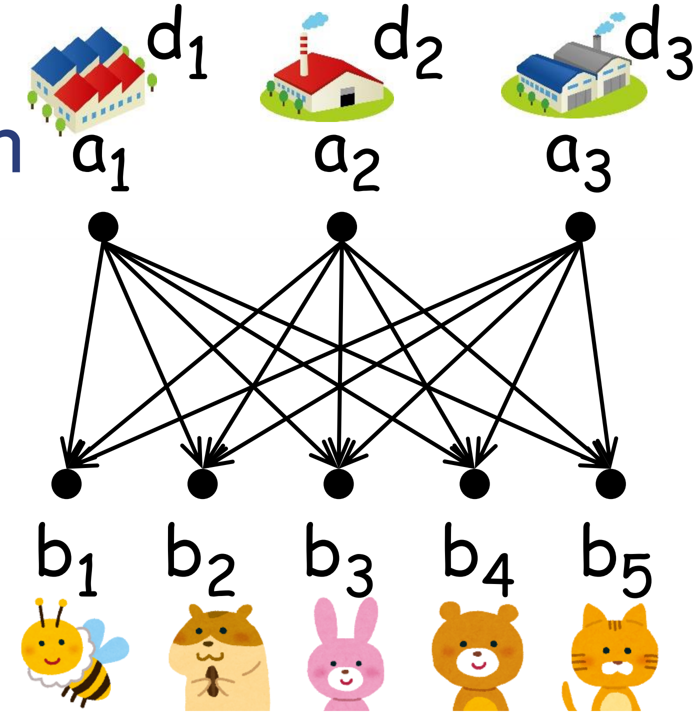
- minimize $\sum_{i=1}^{3} \sum_{j=1}^{5} c_{ij} x_{ij}$

subject to $\sum_{j=1}^{5} x_{ij} \leqq a_i \quad (i = 1, 2, 3)$ Factory i

$\sum_{i=1}^{3} x_{ij} \geqq b_j \quad (j = 1, 2, ..., 5)$ Consumer j

$x_{ij} \geqq 0 \quad (i = 1, 2, 3, \; j = 1, 2, ..., 5)$

Facility location problem



- In the problem in p. 9
  we need cost d_i
  for running factory i

  (In case we do not use factory i,
  its cost becomes 0)

- We need to minimize the sum
  of the transportation cost
  and the running cost of the factories

Formulation of facility location problem

- x_{ij}: the amount from factory i to consumer j
  y_i : run factory i, or not
  ($y_i = 1$ ⋯ run factory i, $y_i = 0$ ⋯ do not use factory i)

- minimize $\sum_{i=1}^{3} \sum_{j=1}^{5} c_{ij} x_{ij} + \sum_{i=1}^{3} d_i y_i$

Factory i $\sum_{j=1}^{5} x_{ij} \leqq a_i y_i$

$$\begin{cases} \sum_{j=1}^{5} x_{ij} \leqq a_i & \text{Run factory } i \ (y_i = 1) \\ \sum_{j=1}^{5} x_{ij} = 0 & \text{Do not use factory } i \ (y_i = 0) \end{cases}$$

Formulation of facility location problem

- x_{ij}: the amount from factory i to consumer j
  y_i : run factory i, or not
  (y_i = 1 ··· run factory i, y_i = 0 ··· do not use factory i)

- minimize $$\sum_{i=1}^{3}\sum_{j=1}^{5} c_{ij} x_{ij} + \sum_{i=1}^{3} d_i y_i$$

  subject to $$\sum_{j=1}^{5} x_{ij} \leqq a_i y_i \quad (i = 1, 2, 3)$$ Factory i

  $$\sum_{i=1}^{3} x_{ij} \geqq b_j \quad (j = 1, 2, ..., 5)$$ Consumer j

  $$x_{ij} \geqq 0 \quad (i = 1, 2, 3,\ j = 1, 2, ..., 5)$$

  $$y_i \in \{0, 1\} \quad (i = 1, 2, 3)$$

Linear programming

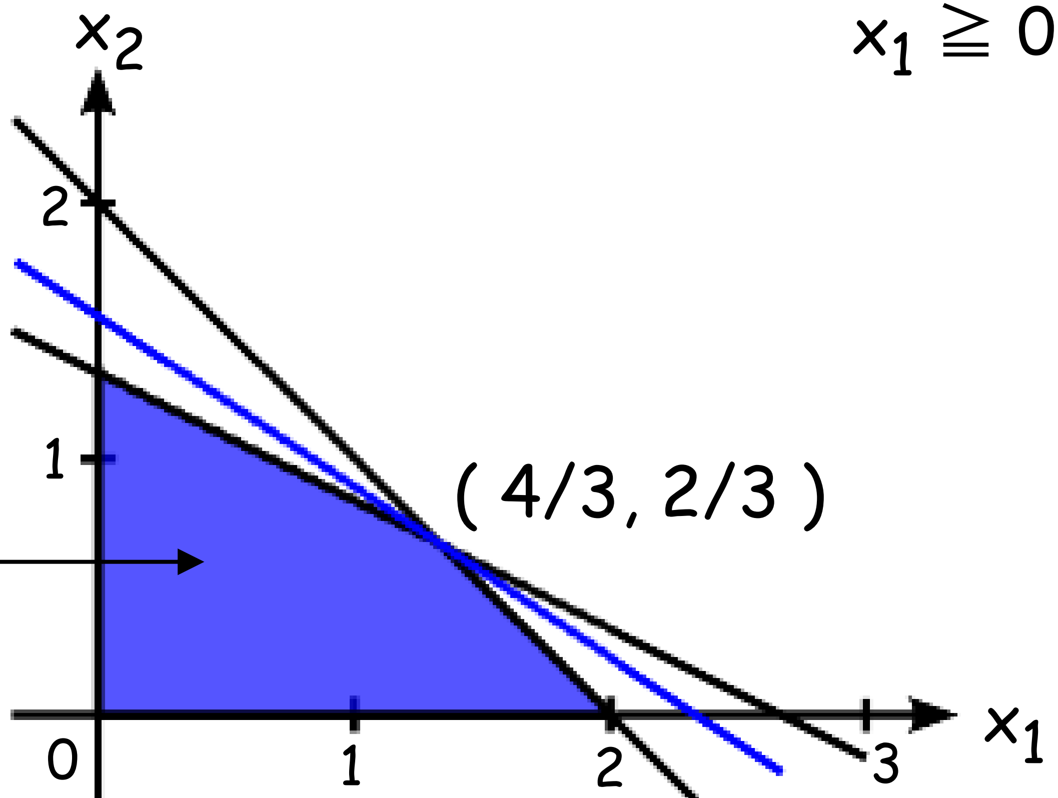
- How to solve linear programming problem ?

Practice: Solve by drawing

- Optimal solution ?
  $(x_1, x_2) =$
- Optimal value ?

- maximize $4x_1 + 5x_2$
  subject to $2x_1 + 2x_2 \leqq 4$
  $3x_1 + 6x_2 \leqq 8$
  $x_1 \geqq 0,\ x_2 \geqq 0$

Feasible region

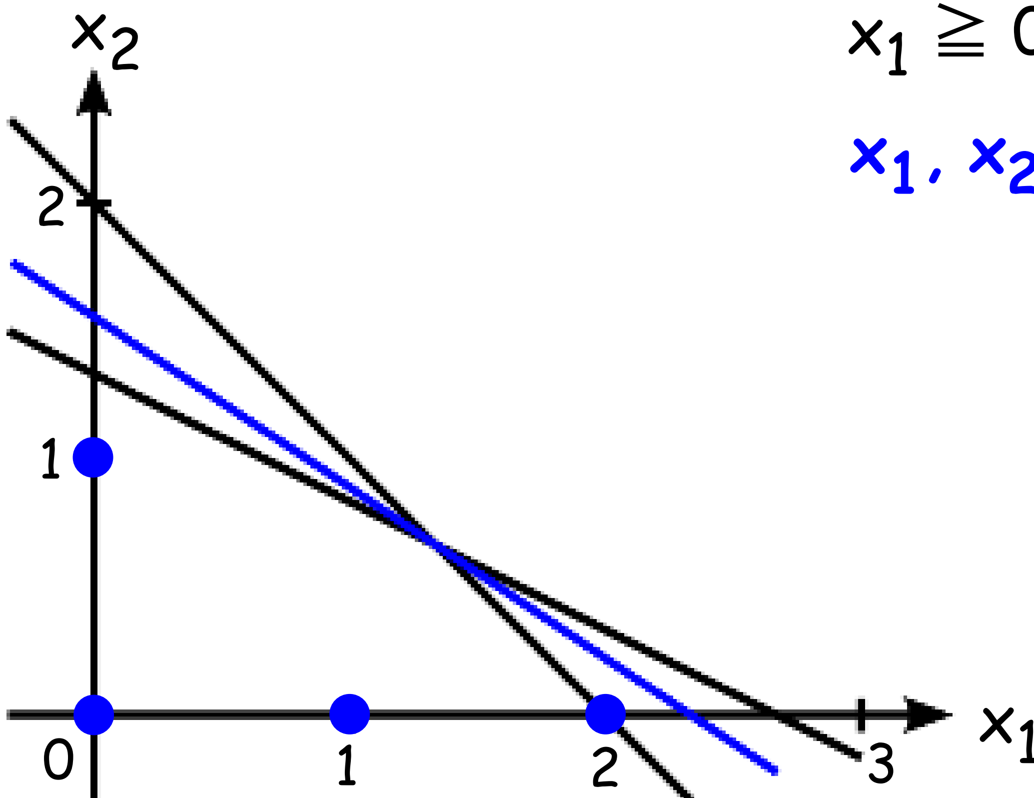


Practice: Solve by drawing (Integer programming)

- Optimal solution ?
  $(x_1, x_2) =$
- Optimal value ?

- maximize $4x_1 + 5x_2$
  subject to $2x_1 + 2x_2 \leqq 4$
  $3x_1 + 6x_2 \leqq 8$
  $x_1 \geqq 0,\ x_2 \geqq 0$
  x_1, x_2 are integers

Feasible set



Practice: Solve by drawing

(a) maximize $4x_1 + 5x_2$

subject to

$$2x_1 + 2x_2 \leqq 4$$
$$3x_1 + 6x_2 \leqq 8$$
$$x_1 + 4x_2 \leqq 4$$
$$x_1 \geqq 0,\ x_2 \geqq 0$$

(b) Change prob. (a) to minimization problem

(c) maximize $4x_1 + 2x_2$

subject to

$$x_1 + x_2 \leqq 8$$
$$x_1 \leqq 6$$
$$x_1 + 2x_2 \geqq 2$$
$$x_1 \geqq 0,\ x_2 \geqq 0$$

(d) Change prob. (c) to minimize problem

Advanced: Solve by drawing (let's challenge)

- Solve the linear programming problem in slide 7
  - **3-dimensional space**
  - $4 x_1 + 2 x_2 + x_3 \leqq 6$
    … half space cut by a plane

- It is very difficult to solve linear programming problems with 3 (or more) variables by drawing …

Linear programming Standard form

Standard form

- We can translate all linear programming problems into their **standard forms**

- **minimize** $z = \sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i \quad (i = 1, 2, \ldots, m)$

$x_j \geqq 0 \quad (j = 1, 2, \ldots, n)$

ただし、$b_i \geqq 0$ (※ Do not forget this)

Ex.) Transformation

- maximize $z = 2x_1 - 3x_2 - 4x_3$

  subject to
  $$2x_1 + x_2 - 3x_3 \leqq 4$$
  $$x_1 - x_2 + 4x_3 \geqq 5$$
  $$x_1, x_2 \geqq 0, \quad x_3: \text{free variable}$$

- maximize $z = 2x_1 - 3x_2 - 4x_3$

  Use "minimize"

- minimize $z = -2x_1 + 3x_2 + 4x_3$

Ex.) Transformation (cont.)

- maximize $z = 2x_1 - 3x_2 - 4x_3$
  subject to
  $2x_1 + x_2 - 3x_3 \leqq 4$
  $x_1 - x_2 + 4x_3 \geqq 5$
  $x_1, x_2 \geqq 0$, x_3: free variable

- $2x_1 + x_2 - 3x_3 \leqq 4$
- $2x_1 + x_2 - 3x_3 + s_1 = 4$, $s_1 \geqq 0$

introduce a **slack variable**

Ex.) Transformation (cont.)

- maximize $z = 2x_1 - 3x_2 - 4x_3$

  subject to $2x_1 + x_2 - 3x_3 \leqq 4$

  $x_1 - x_2 + 4x_3 \geqq 5$

  $x_1, x_2 \geqq 0$, x_3: free variable

- $2x_1 + x_2 - 3x_3 \leqq 4$
- $2x_1 + x_2 - 3x_3 + s_1 = 4$, $s_1 \geqq 0$

introduce a **slack variable**

- $x_1 - x_2 + 4x_3 \geqq 5$
- $x_1 - x_2 + 4x_3 - s_2 = 5$, $s_2 \geqq 0$

introduce a **surplus variable**

Both variables are called **slack variables** in many cases

Ex.) Transformation (cont.)

Both variables are called **slack variables** in many cases

- maximize $z = 2x_1 - 3x_2 - 4x_3$
  subject to $2x_1 + x_2 - 3x_3 \leqq 4$
  $x_1 - x_2 + 4x_3 \geqq 5$
  $x_1, x_2 \geqq 0$, x_3: free variable

- $2x_1 + x_2 - 3x_3 \leqq 4$
- $2x_1 + x_2 - 3x_3 + s_1 = 4$, $s_1 \geqq 0$

introduce a **slack variable**

- $x_1 - x_2 + 4x_3 \geqq 5$
- $x_1 - x_2 + 4x_3 - s_2 = 5$, $s_2 \geqq 0$

introduce a **surplus variable**

- $2x_1 + x_2 - 3x_3' + 3x_3'' + s_1 = 4$
- $x_1 - x_2 + 4x_3' - 4x_3'' - s_2 = 5$, $x_3', x_3'' \geqq 0$

introduce **non-negative** variables

Ex.) Transformation (summary)

- maximize $z = 2x_1 - 3x_2 - 4x_3$

  subject to
  $$2x_1 + x_2 - 3x_3 \leqq 4$$
  $$x_1 - x_2 + 4x_3 \geqq 5$$
  $$x_1, x_2 \geqq 0, \quad x_3\text{: free variable}$$

- minimize $z = -2x_1 + 3x_2 + 4x_3' - 4x_3''$

  subject to
  $$2x_1 + x_2 - 3x_3' + 3x_3'' + s_1 = 4$$
  $$x_1 - x_2 + 4x_3' - 4x_3'' - s_2 = 5$$
  $$x_1, x_2, x_3', x_3'', s_1, s_2 \geqq 0$$

practice : Standard form

(a) maximize $z = 4x_1 + 2x_2$

subject to

$2x_1 + 2x_2 \leqq 4$

$-3x_1 - 6x_2 \leqq -9$

$x_1 \geqq 0$, x_2: free variable

(b) maximize $z = 3x_1 + 5x_2 + 4x_3$

subject to

$4x_1 + 2x_2 + 3x_3 \leqq 6$

$3x_1 - 4x_2 + 5x_3 \leqq -2$

$x_1 \geqq 0$, $x_2 \geqq 0$, x_3: free variable

Summary

- Formulation of the problems
  - **Linear programming problem**
  - **Integer programming problem**
- Solve linear programming problems
  - Solve by drawing
  - It is very difficult to solve linear programming problems with 3 (or more) variables
- **Standard form** of a linear programming problem
  - Any linear programming problem can be transformed into its standard form