Large-Scale Knowledge Processing #5

Optimization Techniques (Part 2)

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Prev. class + α : Linear Programming Problem Standard form

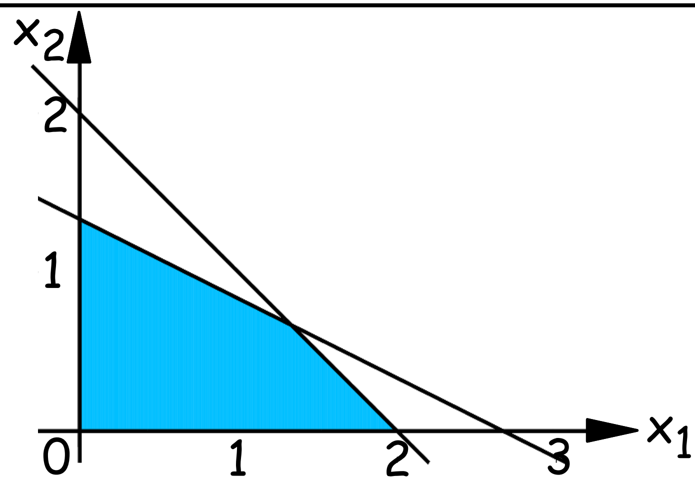
- minimize $z = -4x_1 - 5x_2$
  subject to
  $$2x_1 + 2x_2 + x_3 = 4$$
  $$3x_1 + 6x_2 + x_4 = 8$$
  $$x_1, x_2, x_3, x_4 \geqq 0$$

Transformation to its standard form

maximize $z = 4x_1 + 5x_2$
subject to
$$2x_1 + 2x_2 \leqq 4$$
$$3x_1 + 6x_2 \leqq 8$$
$$x_1, x_2 \geqq 0$$

Intuitive observation :

- By checking the **extreme points** (vertices) of the feasible region, we can find the optimal solution
- How can we find **extreme points** of the feasible region ? (The standard form gives a good hint)



Ex.) Standard form (Vector representation)

- minimize $z = -4x_1 - 5x_2$

  subject to
  $$2x_1 + 2x_2 + x_3 = 4$$
  $$3x_1 + 6x_2 + 4x_4 = 8$$
  $$x_j \geqq 0 \quad (j = 1, 2, \ldots, 4)$$

- minimize
  $$z = (-4 \quad -5 \quad 0 \quad 0)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

subject to
$$\begin{pmatrix} 2 & 2 & 1 & 0 \\ 3 & 6 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geqq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Standard form (Vector representation)

$$A = \begin{pmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$$

- minimize $\quad z = \sum_{j=1}^{n} c_j x_j$

  subject to $\quad \sum_{j=1}^{n} a_{ij} x_j = b_i \qquad (i = 1, 2, \ldots, m)$

  $\qquad x_j \geqq 0 \qquad (j = 1, 2, \ldots, n)$

- minimize $\quad z = c^T x$

  subject to $\quad A x = b$

  $\qquad x \geqq 0$

$c^T = (c_1, c_2, \ldots, c_n)$

$A = (a_{ij}) \quad m \times n$ matrix

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Standard form (Vector representation)

$$A = \begin{pmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

A can be regarded as $(A_1 \ A_2 \cdots A_n)$

(i.e., concatenation of n vertical vectors)

- minimize $z = c^T x$
  subject to $A x = b$
  $x \geqq 0$

$c^T = (c_1, c_2, \ldots, c_n)$

$A = (a_{ij})$ $m \times n$ matrix

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Ex.) Basis

Constraints in p. 3

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 3 & 6 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

- Infinitely many assignments to (x_1, x_2, x_3, x_4) satisfying the 2 conditions
  - In case $(x_3, x_4) = (0, 0)$?
  - i.e., if we **focus on x_1, x_2**

    → (x_1, x_2, x_3, x_4) is uniquely obtained

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 3 & 6 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 2/3 \end{pmatrix}$$

  - **Basic vectors**: $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix} \right\}$
  - **Basic matrix**: $\begin{pmatrix} 2 & 2 \\ 3 & 6 \end{pmatrix}$
  - **Basic variables**: x_1, x_2, **Nonbasic variables**: x_3, x_4
  - **Basic solution**

    for basic vectors $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix} \right\}$: $\hat{x} = (4/3, 2/3, 0, 0)$

Ex.) Basis

Constraints in p. 3

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 3 & 6 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

- Try another base vector

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 3 & 6 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 8/3 \\ -4/3 \end{pmatrix}$$

- **Basic vectors**: $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$
- **Basic matrix**: $\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$
- **Basic variables**: x_1, x_3, **Nonbasic variables**: x_2, x_4
- **Basic solution**
  for basic vectors $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$: $\hat{x} = (8/3, 0, -4/3, 0)$

Supplementary information

- Some basic matrix **may not have the inverse matrix**

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$ are linear dependent (not independent)

- $$A = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$
  - Assumption 1: $n \geqq m$
  - Assumption 2: the rank of A is m
  - Assumption 3: optimal solution exists

We can relax these assumptions (later)

Basic vector
Basic matrix

$$A = \begin{pmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} & \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} & \cdots & \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \end{pmatrix} = (A_1 \quad A_2 \quad \cdots \quad A_n)$$

- **Basic vector**

  Their combination is not unique

  - **m** linear independent vectors in A

    Basis $\mathcal{B} = \{ A_{i_1}, A_{i_2}, \ldots, A_{i_m} \}$

$$\begin{pmatrix} 2 & 2 & 1 & 0 \\ 3 & 6 & 0 & 4 \end{pmatrix}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix} \right\}$$

- **Non basic vector**
  - Other **(n - m)** vectors

- **Basic matrix** B
  - m x m square matrix with **m** basic vectors

    $B = (A_{i_1} \quad A_{i_2} \cdots A_{i_m})$

$$\begin{pmatrix} 2 & 2 \\ 3 & 6 \end{pmatrix}$$

- **Nonbasic matrix** N
  - m x (n - m) matrix with **(n - m)** Nonbasic vectors

Basic variable
Basic solution

$x_B = (x_1, x_2)$

$x_N = (x_3, x_4)$

$\hat{x} = (8/3, 2/3, 0, 0)$

- **Basic variable**
  - **m** basic variables $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$ corresponding to m basic vectors $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$
  - m-dimensional vector $x_B = (x_{i_1}, x_{i_2}, \ldots, x_{i_m})$
- **Nonbasic variable**
  - Other **(n−m)** variables
  - (n−m)-dimensional vector x_N
- **Basic solution $\hat{x}$** for basis $\mathcal{B} = \{ A_{i_1}, A_{i_2}, \ldots, A_{i_m} \}$
  - $$x_j = \begin{cases} 0 & (A_j \notin \mathcal{B}) \\ \ell\text{-th element of } B^{-1}b & (A_j \in \mathcal{B}, x_j = x_{i_\ell}) \end{cases}$$

Solution of Bx = b

Standard form in p. 3

practice: Basic solution

- minimize $z = -4x_1 - 5x_2$

  subject to
  $$2x_1 + 2x_2 + x_3 = 4$$
  $$3x_1 + 6x_2 + x_4 = 8$$
  $$x_j \geqq 0 \quad (j = 1, 2, \ldots, 4)$$

- $A_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $A_2 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$, $A_3 = (\quad)$, $A_4 = (\quad)$
- A_1, A_2 independent : $\mathrm{rank}(A_1 A_2) = m$ (determinant≠0)
- Solve $(A_1 \; A_2)\begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$: we have $x_1 =$, $x_2 =$
- Basic solution for A_1, A_2 is $x^T = (\quad,\quad,\quad,\quad)$
- All basic elements (elements of basic solution) are non-negative → **basic feasible solution**

practice： Basic solution

- From the conditions of the problem in p. 3, we have 4 vectors $A_1 \sim A_4$. We have 6 ways for choosing 2 basic vectors A_i and A_j. For each of them, find basic solutions.
- From each of the above basic solutions, obtain the coordinate (x_1, x_2). Find such coordinates in the graph of the original optimization problem.

Original optimization prob.

$$
\begin{aligned}
\text{maximize} \quad & 4x_1 + 5x_2 \\
\text{subject to} \quad & 2x_1 + 2x_2 \leqq 4 \\
& 3x_1 + 6x_2 \leqq 8 \\
& x_1 \geqq 0,\ x_2 \geqq 0
\end{aligned}
$$

Standard form in p. 3

Standard form, basic solution

$Ax = b$
$x \geqq 0$

- minimize $z = -4x_1 - 5x_2$
  subject to
  $2x_1 + 2x_2 + x_3 = 4$
  $3x_1 + 6x_2 + x_4 = 8$
  $x_1, x_2, x_3, x_4 \geqq 0$

$A_1 \; A_2 \; A_3 \; A_4$

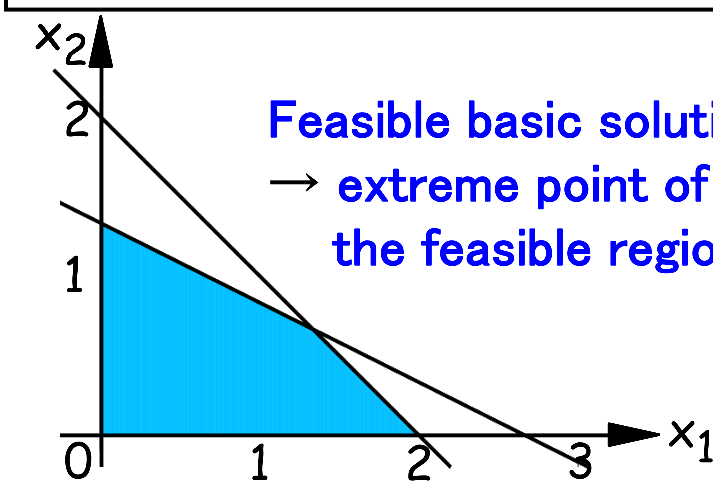
$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Transformation to its standard form

maximize $z = 4x_1 + 5x_2$
subject to
$2x_1 + 2x_2 \leqq 4$
$3x_1 + 6x_2 \leqq 8$
$x_1, x_2 \geqq 0$

basic solutions for

- A_1, A_2 $\quad (\frac{4}{3}, \frac{2}{3}, 0, 0)$
- A_1, A_3 $\quad (\frac{8}{3}, 0, -\frac{4}{3}, 0)$
- A_1, A_4 $\quad (2, 0, 0, 2)$
- A_2, A_3 $\quad (0, \frac{4}{3}, \frac{4}{3}, 0)$
- A_2, A_4 $\quad (0, 2, 0, -4)$
- A_3, A_4 $\quad (0, 0, 4, 8)$



Feasible basic solution → extreme point of the feasible region

Degeneration of feasible basic solutions

- Feasible basic solution x is **degenerate**
  - x has more 0's than n−m
- Two different basis correspond to the same basic solution x

  → x is degenerate

practice : Degeneration

- For the problems in Slide #3 p. 20 (a), p. 21
  (1) Transform them into their standard forms
  (2) Check whether they have degenerate feasible basic solution

Degeneration of feasible basic solutions

- minimize $z = -4x_1 - 5x_2$

  subject to
  $$\begin{aligned} 2x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 + 6x_2 + x_4 &= 8 \\ x_1 + 4x_2 + x_5 &= 4 \\ x_1, x_2, x_3, x_4, x_5 &\geqq 0 \end{aligned}$$

$$A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5$$
$$\begin{pmatrix}2\\3\\1\end{pmatrix}\begin{pmatrix}2\\6\\4\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

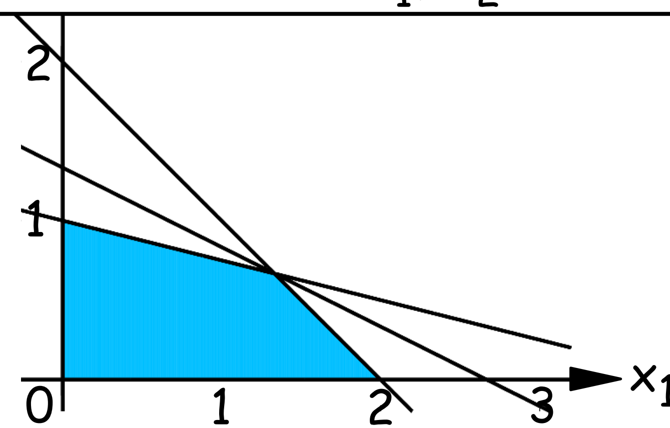
maximize $z = 4x_1 + 5x_2$

subject to
$$\begin{aligned} 2x_1 + 2x_2 &\leqq 4 \\ 3x_1 + 6x_2 &\leqq 8 \\ x_1 + 4x_2 &\leqq 4 \\ x_1, x_2 &\geqq 0 \end{aligned}$$

basic solutions for

- A_1, A_2, A_3 $\left(\frac{4}{3}, \frac{2}{3}, 0, 0, 0\right)$
- A_1, A_2, A_4 $\left(\frac{4}{3}, \frac{2}{3}, 0, 0, 0\right)$
- A_1, A_2, A_5 $\left(\frac{4}{3}, \frac{2}{3}, 0, 0, 0\right)$

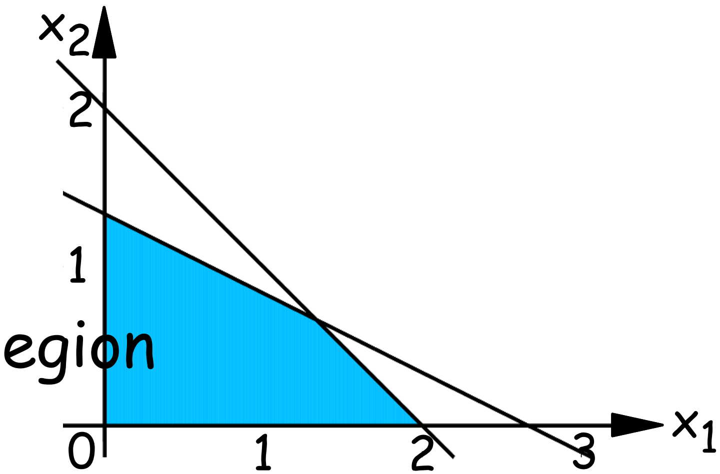
⋮



practice : Basic solution

- Enumerate all basic solutions for the problem in p. 15
  - We have 5 basic vectors A_1, A_2, A_3, A_4, A_5
  - Choose any 3 from these 5

Summary (1st half)



- Feasible basic solution → extreme point of feasible region
- Basis → basic solution
- We can find the optimal solution by checking **all** basic solutions for **all** basis

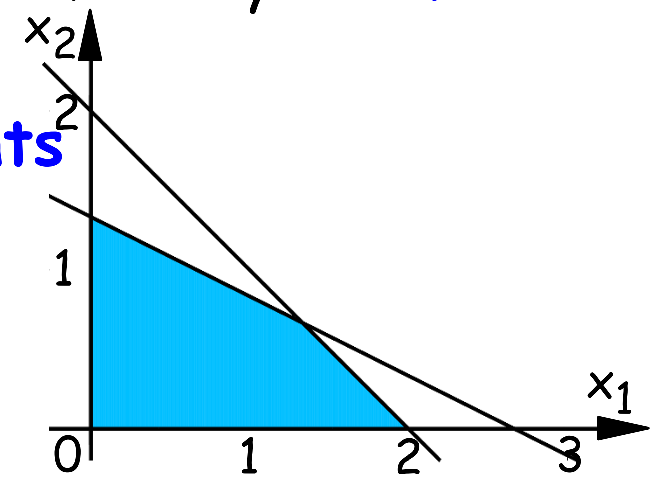
Linear programming
Simplex method (Simplex algorithm)

- Check all basic solutions ?
- Better ways ?

  Walk through feasible basic solutions with improving the objective value

preliminaries: Feasible region, extreme point

- Lemma: Feasible region F is convex
- Definition: $x \in F$ is an **extreme point**
  - The point that cannot be a convex combination of two different points $x', x'' \in F$
- Properties:
  - The number of **extreme point** for any F is **finite**
  - **F** can be represented as a **convex hull** of **extreme points**



- S is convex
  - $\forall\, x', x'' \in S,\ \forall\, \alpha\ (0 \leqq \alpha \leqq 1)$
    $x' + (1-\alpha)x'' \in S$
- Convex hull of point set S'
  - Minimum convex region containing S'

Simplex dictionary (Dictionary)

- minimize $z = -2x_1 + x_2 + x_3 - x_4$
  subject to $x_1 + 2x_2 - x_3 + x_4 = 0$
  $2x_1 - 2x_2 + 3x_3 + 4x_4 = 9$
  $x_1, x_2, x_3, x_4 \geqq 0$

- Basic solution for $A_2\ A_3$

$\left(0, \frac{9}{4}, \frac{9}{2}, 0\right)$ ··· feasible solution

- Dictionary
  - Solve the conditions for **basic variables**
    + substitute them in the **objective function**

Current objective value

$$z = 27/4 - 19/4\, x_1 - 21/4\, x_4$$
$$x_2 = 9/4 - 5/4\, x_1 - 7/4\, x_4$$
$$x_3 = 9/2 - 3/2\, x_1 - 5/2\, x_4$$

Basic variables

Values in basic solution

Nonbasic variables

Basis exchange (Pivot operation)

- $z = 27/4 - 19/4\, x_1 - 21/4\, x_4$
- $x_2 = 9/4 - 5/4\, x_1 - 7/4\, x_4$
- $x_3 = 9/2 - 3/2\, x_1 - 5/2\, x_4$

- $x_1 = x_4 = 0$ holds in the current basic solution
- The coefficients of x_1, x_4 in the objective function are $-19/4$, $-21/4$, respectively
- If we increase x_4 by Δ
  - the objective value decreases: $27/4 - 21/4\, \Delta$
  - If x_4 increases by itself (other variables are fixed), the constraints are not satisfied

    → **change basic variables** (x_2, x_3)
    with fixing other nonbasic variables ($x_1 = 0$)

Basis exchange (Pivot operation)

- $z = 27/4 - 21/4\, x_4$
- $x_2 = 9/4 - 7/4\, x_4$
- $x_3 = 9/2 - 5/2\, x_4$

- **Fix other nonbasic variables** ($x_1 = 0$)
- Increase x_4 from 0
  - x_2 decreases (when $x_4 = 9/7$, we have $x_2 = 0$)
  - x_3 decreases (when $x_4 = 9/5$, we have $x_3 = 0$)

- Stop at $x_4 = 9/7$ to keep the **feasibility** ($x_i \geqq 0$)
  - $x_2 = 0$, i.e., "x_2 becomes a nonbasic variable"
  - x_4 becomes a basic variable

Basis exchange (Pivot operation)

- $z = 27/4 - 19/4\, x_1 - 21/4\, x_4$
- $x_2 = 9/4 - 5/4\, x_1 - 7/4\, x_4$
- $x_3 = 9/2 - 3/2\, x_1 - 5/2\, x_4$

Update a dictionary

- x_2 becomes a nonbasic variable,
  x_4 becomes a basic variable
- Transform eq. $x_2 = \ldots$ as eq. $x_4 = \ldots$
  & Substitute x_4 in other equations

- $z = -9/4 + 19/5\, x_1 + 7/5\, x_2$
- $x_4 = 9/7 - 5/7\, x_1 - 4/7\, x_2$
- $x_3 = 9/5 + 6/5\, x_1 - 2/5\, x_2$

practice: Pivot operation

- Update the following simplex dictionary so that x_2, x_3 are basic variables and x_1, x_4 are nonbasic variables

$$
\begin{aligned}
z &= \quad -4x_1 - 6x_2 \\
x_3 &= 4 - 2x_1 - 2x_2 \\
x_4 &= 9 - 3x_1 - 6x_2
\end{aligned}
$$

Optimal dictionary

- All coefficients (of nonbasic variables) in the objective function are positive
  - We cannot decrease objective value

- **For all feasible solution,**
  $x_2, x_4 \geqq 0$ implies $z \geqq -9/4$
- $x^T = (9/5, 0, 9/5, 0)$ is **optimal** with $z = -9/4$

Optimal dictionary

Optimal value

Optimal solution $x^T = (9/5, 0, 9/5, 0)$

- $z = -9/4 + 19/5\, x_2 + 7/5\, x_4$
- $x_1 = 9/5 - 4/5\, x_2 - 7/5\, x_4$
- $x_3 = 9/5 + 6/5\, x_2 - 2/5\, x_4$

Simplex tableau

$$
\begin{aligned}
z &= 27/4 - 19/4\, x_1 - 21/4\, x_4 \\
x_2 &= 9/4 - 5/4\, x_1 - 7/4\, x_4 \\
x_3 &= 9/2 - 3/2\, x_1 - 5/2\, x_4
\end{aligned}
$$

- Represent a dictionary by a **simplex tableau**

		x_1	x_4
z	27/4	- 19/4	- 21/4
x_2	9/4	- 5/4	- 7/4
x_3	9/2	- 3/2	- 5/2

Current objective function: 27/4

Basic variables: x_2, x_3

Nonbasic variables: x_1, x_4

Values in basic solution: 9/4, 9/2

Relative cost coefficients: - 19/4, - 21/4

Rates of change of the objective value with respect to the nonbasic variables

- **Pivot operation**
  - Pivot (r, s) : r-th row & s-th column
  - Exchange the r-th basic var. & the s-th nonbasic var.
  - Ex.: Pivot (1, 1) to the above tableau
    - Transform eq. x_2 = … as eq. x_1 = … & Substitute x_1 in other equations

x_1: Pivot in variable (Variable entering the basis)

x_2: Pivot out variable (Variable exiting from the basis)

Simplex method

		x_1	x_2
z	0	- 4	- 6
x_3	4	- 2	- 2
x_4	9	- 3	- 6

minimize $z = -4\,x_1 - 6\,x_2$

subject to

$$2\,x_1 + 2\,x_2 + x_3 = 4$$
$$3\,x_1 + 6\,x_2 + x_4 = 9$$
$$x_1, x_2, x_3, x_2 \geqq 0$$

		x_1	x_4
z	-9	- 1	1
x_3	1	- 1	1/3
x_2	3/2	- 1/2	- 1/6

Pivot (2, 2)

Pivot the 2nd row and 2nd column

z		

Pivot (,   )

Summary (2nd half)

- Represent a feasible basic solution (an extreme point of the feasible region) by a simplex tableau
- Overview of Simplex method
  - Find a starting point:
    Find a feasible basic solution (given in the next class)
  - Pivot operation:
    Exchange the r-th basic var. & the s-th nonbasic var.
    → Walk through the extreme points
    (until we cannot improve the objective function)