Large-Scale Knowledge Processing #6 Optimization Techniques (Part 3)

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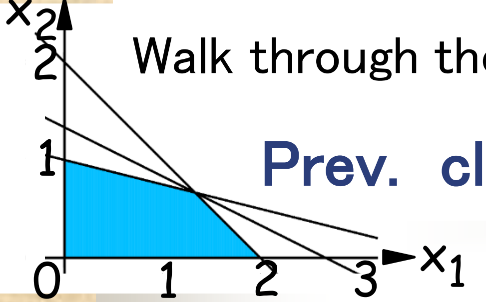


Walk through the extreme points from an initial feasible basic solution

Prev. class + α : Simplex method

$$
\begin{aligned}
\text{mimimize} \quad & z = -4x_1 - 5x_2 \\
\text{subject to} \quad & 2x_1 + 2x_2 + x_3 = 4 \\
& 3x_1 + 6x_2 + x_4 = 8 \\
& x_1 + 4x_2 + x_5 = 4 \\
& x_1, x_2, x_3, x_4, x_5 \geqq 0
\end{aligned}
$$

Initial feasible basic solution

		x_1	x_2
z	0	- 4	- 5
x_3	4	- 2	- 2
x_4	8	- 3	- 6
x_5	4	- 1	- 4

$$
\begin{aligned}
z &= -4x_1 - 5x_2 \\
x_3 &= -2x_1 - 2x_2 + 4 \\
x_4 &= -3x_1 - 6x_2 + 8 \\
x_5 &= -x_1 - 4x_2 + 4
\end{aligned}
$$

Pivot (3, 2)

Increase x_2 from 0
(Fix other nonbasic var. $x_1 = 0$)

x_3 decrease, $x_3 \geqq 0$ if $x_2 \leqq 2$
x_4 decrease, $x_4 \geqq 0$ if $x_2 \leqq 4/3$
x_5 decrease, $x_5 \geqq 0$ if $x_2 \leqq 1$

$$x_2 = -1/4\, x_1 - 1/4\, x_5 + 1$$

		x_1	x_5
z	-5	-11/4	5/4
x_3	2	- 3/2	1/2
x_4	2	- 3/2	- 3/2
x_2	1	- 1/4	- 1/4

Pivot (1, 1)

		x_3	x_5
z	-26/3	11/6	1/3
x_1	4/3	- 2/3	1/3
x_4	0	1	1
x_2	2/3	1/6	- 1/3

Initial **feasible** basic solution ?

- General way will be given later
- Easy case:

maximize $z = 4x_1 + 6x_2$ subject to $2x_1 + 2x_2 \leqq 4$ $3x_1 + 6x_2 \leqq 9$ $x_1, x_2 \geqq 0$	minimize $z = -4x_1 - 6x_2$ subject to $2x_1 + 2x_2 + x_3 = 4$ $3x_1 + 6x_2 + x_4 = 9$ $x_1, x_2, x_3, x_4 \geqq 0$

Transformation to its standard form

- Basic vars.: x_3, x_4
  - $z = -4x_1 - 6x_2$
  - $x_3 = 4 - 2x_1 - 2x_2$
  - $x_4 = 9 - 3x_1 - 6x_2$

Choice of pivot ?

- One of the typical ways:
  select variable x_j whose coefficient c_j is $c_j < 0$ and $|c_j|$ is the largest

Tips: Pivot operation

just update the simplex tableau

		x_1	x_2
z	0	- 4	- 6
x_3	4	- 2	- 2
x_4	9	- 3	- 6

		x_1	x_4
z			
x_3			
x_2	3/2	- 1/2	- 1/6

9 / -(- 6)

1 / (- 6)

Ex.: Pivot (2, 2)

Exchange the var. in the **2nd row** and the var. in the 2nd column

2nd row

$x_4 = b_4 + a_1 x_1 + a_2 x_2$

$x_2 = b_4/(-a_2) + a_1/(-a_2)\, x_1 + 1/a_2\, x_4$

Divide all elements in the **2nd row** by $-a_2$, but $1/a_2$ in the 2nd column

Tips: Pivot operation

just update the simplex tableau

		x_1	x_2
z	0	- 4	- 6
x_3	4	- 2	- 2
x_4	9	- 3	- 6

		x_1	x_4
z			
x_3			
x_2	3/2	- 1/2	- 1/6

$4 + (3/2)(-2)$

		x_1	x_4
z			
x_3	1	- 1	1/3
x_2	3/2	- 1/2	- 1/6

$(-1/6)(-2)$

Ex.: Pivot (**2**, 2)

Exchange the var. in the **2nd row** and the var. in the 2nd column

Other rows

$x_3 = b_3 + a_1 x_1 + \underline{a_2 x_2}$

$x_2 = b'_2 + a'_1 x_1 + a'_4 x_4$

↓

(Eq. on x_2) $\times a_2$ + (Eq. on x_3)

i.e., i-th column becomes $a_i a_2 + a'_i$

Notice: 2nd column: (Eq. x_2) $\times a_2$

Practice: Simplex method

Solve the following linear programming problems by simplex method

a. The problem in lecture #5 slide p. 24
b. #5 p. 13
c. #5 p. 15
d. #3 p. 7

Linear programming
Simplex method

- Can we solve all linear programming problem？

Situation 1 : Feasible region is unbounded

Original problem

mimimize $z = -x_1 - 2x_2$
subject to $x_1 + x_2 \geqq -1$
$x_1, x_2 \geqq 0$

Initial feasible basic solution

		x_1	x_2
z	0	- 1	- 2
x_3	1	1	1

Standard form

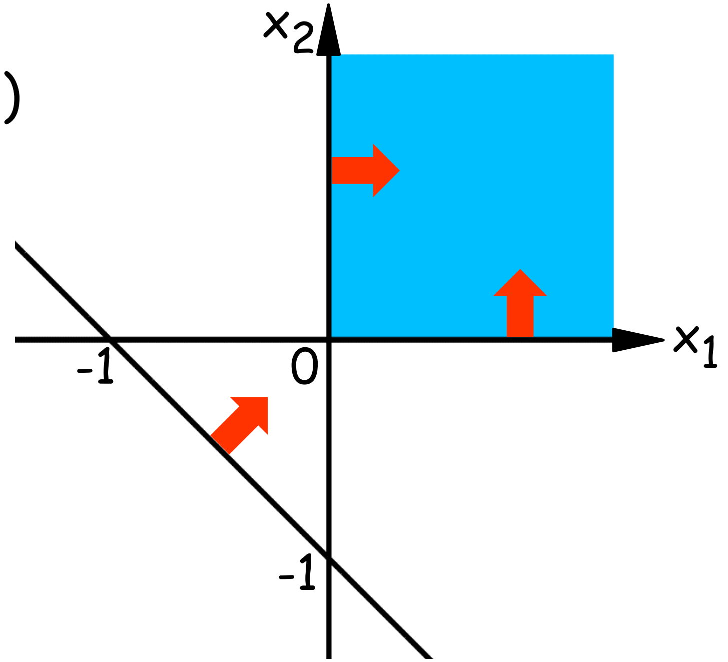
$-x_1 - x_2 + x_3 = 1$
$x_1, x_2, x_3 \geqq 0$

Increase x_2 from 0
(Fix other nonbasic variables as $x_1 = 0$)

x_3 increases, always satisfies $x_3 \geqq 0$

→ Objective value z decreases

Feasible region is **unbounded**



Situation 2: Infeasible (no feasible solutions)

Original problem

mimimize $z = -x_1 - 2x_2$
subject to $x_1 + x_2 \leqq -1$
$x_1, x_2 \geqq 0$

Initial feasible basic solution

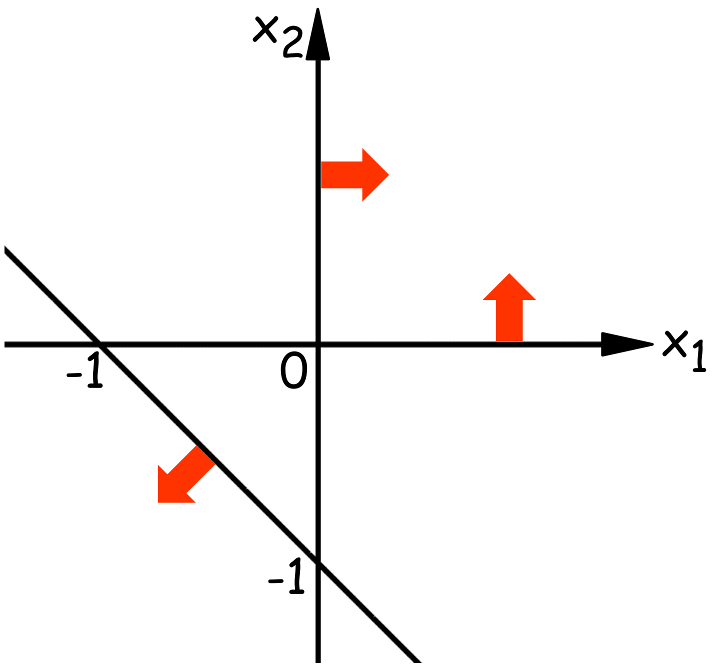
		x_1	x_2
z	0	- 1	- 2
x_3	- 1	- 1	- 1

Not a feasible basic solution

Standard form

$-x_1 - x_2 - x_3 = 1$
$x_1, x_2, x_3 \geqq 0$

How to decide the problem is infeasible ???



Situation 3: Initial feasible basic solution

Original problem

minimize $z = -x_1 - 2x_2$
subject to $x_1 + x_2 \geqq 1$
$x_1, x_2 \geqq 0$

Standard form

$x_1 + x_2 - x_3 = 1$
$x_1, x_2, x_3 \geqq 0$

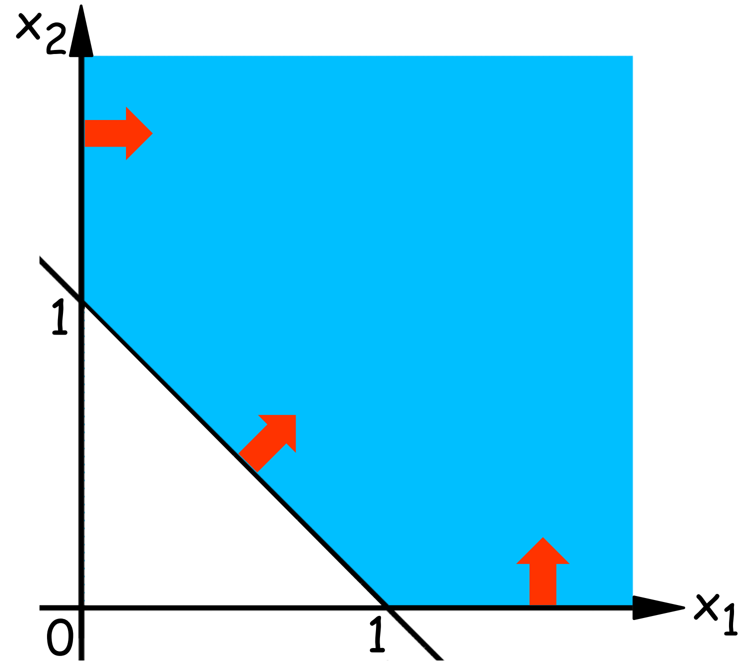
Initial feasible basic solution

		x_1	x_2
z	0	- 1	- 2
x_3	- 1	1	1

Not a feasible basic solution
(The right fig. says (0, 0) is not a feasible solution, but we have feasible region)

		x_1	x_3
z	0	1	- 2
x_2	1	- 1	1

(0, 1) is a feasible basic solution



How to find an initial feasible basic solution ???

Two-phase simplex method

- Original problem (in standard form)

$$\begin{aligned} \text{mimimize} \quad & z = -x_1 - 5x_2 \\ \text{subject to} \quad & 4x_1 - x_2 + 4x_3 = 6 \\ & x_1 + 2x_2 + 2x_3 = 4 \\ & x_1, x_2, x_3 \geqq 0 \end{aligned}$$

How to find an initial feasible basic solution ???

- Artificial problem

$$\begin{aligned} \text{mimimize} \quad & w = x_4 + x_5 \\ \text{subject to} \quad & 4x_1 - x_2 + 4x_3 + x_4 = 6 \\ & x_1 + 2x_2 + 2x_3 + x_5 = 4 \\ & x_1, x_2, x_3, x_4, x_5 \geqq 0 \end{aligned}$$

Introduce an **artificial variable** for each eq.

Minimize the sum of the **artificial vars.**

(so that **all artificial vars.** are **0s.**)

Initial feasible basic solution

$(x_1, x_2, x_3, x_4, x_5)$

$= (0, 0, 0, 6, 4)$

Phase I: apply simplex method to the artificial problem

Optimal solution of the artificial problem

$(0, 2/5, 8/5, 0, 0)$

This solution satisfies the constraints of the original prob.

= This is the initial feasible basic solution for the original prob.

Phase II: apply simplex method to the original prob.

Formulation of linear programming problems

(additional techniques)

max{ } function

- Minimize the maximum of two or more objective functions
  - minimize $\max\{ f(x), g(x), h(x) \}$

  → Introduce a new variable z

  minimize z
  subject to $f(x) \leqq z,\ g(x) \leqq z,\ h(x) \leqq z$

Practice :

- minimize $x_1 + |x_2|$

max{ } function

- Minimize the maximum of two or more objective functions
  - minimize max{ f(x), g(x), h(x) }

→ Introduce a new variable z

minimize z
subject to $f(x) \leqq z,\ g(x) \leqq z,\ h(x) \leqq z$

Practice:

- minimize $x_1 + |x_2|$

$|x_2| = \max\{ x_2, -x_2 \}$

→ minimize $x_1 + x_2'$
subject to $x_2 \leqq x_2',\ -x_2 \leqq x_2'$

Practice： Constraints (Mixed integer programming prob.)

- We have 2 variables x and y, where x = 0, 1, and y is a real number with y ≧ 0. Show constraint inequalities for each of the following constraints

1. y ≧ 30 when x = 1
2. y = 0 when x = 0
3. y ≦ 10 when x = 0
4. y ≦ 10 when x = 0, and y ≧ 30 when x = 1
5. y ≦ 10 when x = 0, and y ≦ 30 when x = 1

Constraints (Mixed integer programming prob.)

1. $y \geqq 30$ when $x = 1$
   - $\mathbf{y \geqq 30\,x}$
     i.e., $\mathbf{-\,30\,x + y \geqq 0}$
   - When $x = 0$, this inequality says $y \geqq 0$ (thus, we have no constraint in this case)
   - When $x = 1$, we add constraint $y \geqq 30$

Constraints (Mixed integer programming prob.)

2. $y = 0$ when $x = 0$
   - $\mathbf{y \leqq M x}$ (M is a sufficiently large constant)
     i.e., $\mathbf{-Mx + y \leqq 0}$
   - When $x = 0$, we have $y \leqq 0$.
     As we have $y \geqq 0$, we have constraint $y = 0$

3. $y \leqq 10$ when $x = 0$
   - $\mathbf{-Mx + y \leqq 10}$ (M is a sufficiently large constant)

Constraints (Mixed integer programming prob.)

4. $y \leqq 10$ when $x = 0$, $y \geqq 30$ when $x = 1$
   - $-Mx + y \leqq 10$, (M is a sufficiently large constant)
     $-30x + y \geqq 0$
   - From the above constraints 1 and 3

5. $y \leqq 10$ when $x = 0$, $y \leqq 30$ when $x = 1$
   - $y \leqq 10(1 - x) + 30x$
     i.e., $-20x + y \leqq 10$

Practice: Formulation

From Naoki Katoh, Mathematical programming, Corona publishing, p. 58【7】

We are given n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the plane. Each point is colored in red or blue.

1. We'd like to obtain a parabola that is convex at the bottom, such that **all red points** are **above** the parabola and **all blue points** are **below** the parabola. Formulate the problem of **finding the equation of such a parabola** (if it exists), as a linear programming problem.

2. We'd like to obtain a circle such that all **red points** are **inside the circle** (including the boundary) and all **blue points** are **outside the circle** (including the boundary) Formulate the problem of **determining** whether such a **circle exists** or not, as a linear programming problem.

Formulation (1)

From Naoki Katoh, Mathematical programming, Corona publishing, p. 58【7】

- Without loss of generality, we have
  k red points $(x_1, y_1), \ldots, (x_k, y_k)$, and
  n – k blue points $(x_{k+1}, y_{k+1}), \ldots, (x_n, y_n)$
- Let $y = a x^2 + b x + c$ denote the parabola
- Formulation is as follows:
  - minimize 0
  - subject to

Determine whether we have a feasible solution (a, b, c) satisfying the constraints

Red points: $x_i^2 a + x_i b + c \geqq y_i \quad (i = 1, 2, \ldots, k)$

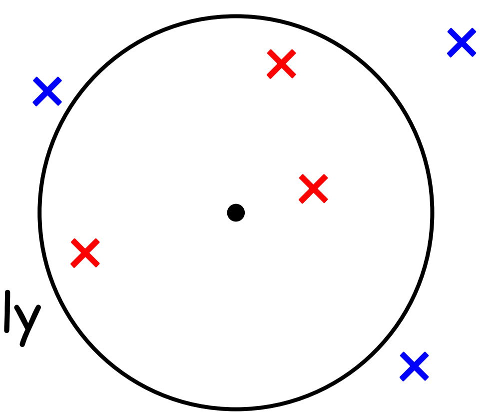
Blue points: $x_i^2 a + x_i b + c \leqq y_i \quad (i = k+1, k+2, \ldots, n)$

$a > 0$, b, c: free variables

Formulation (2)

From Naoki Katoh, Mathematical programming, Corona publishing, p. 58【7】

- Without loss of generality, we have k red points $(x_1, y_1), \ldots, (x_k, y_k)$, and $n - k$ blue points $(x_{k+1}, y_{k+1}), \ldots, (x_n, y_n)$
- Let (x, y) and r denote the center and the radius of the circle, respectively



- Constraints for each point :

Red points : $(x_i - x)^2 + (y_i - y)^2 \leqq r^2 \quad (i = 1, 2, \ldots, k)$

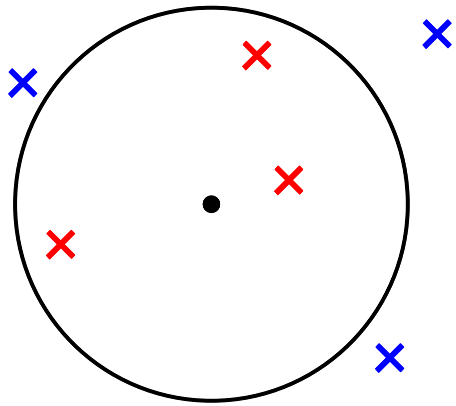
Blue points : $(x_i - x)^2 + (y_i - y)^2 \geqq r^2 \quad (i = k+1, k+2, \ldots, n)$

- As we have x^2, y^2, r^2 in the constraints, we cannot formulate the problem as a linear programming problem ... (is that right ?)

Formulation (2)

From Naoki Katoh, Mathematical programming, Corona publishing, p. 58【7】

- Let point C (x, y) be the center of the circle
- We try to denote the constraints in another way
- Any pair of **red point P_i (x_i, y_i)** and **blue point P_j (x_j, y_j)** satisfies **distance $C P_i \leqq$ distance $C P_j$**



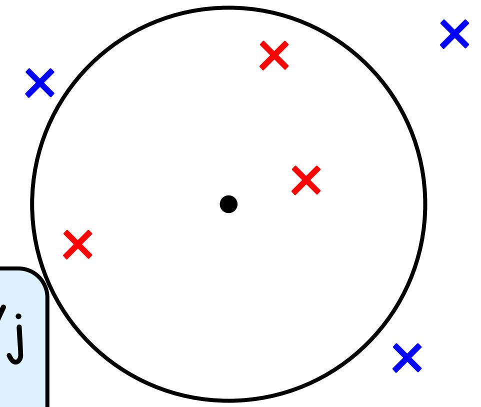
- That is, for any $i = 1, 2, \ldots, k$ and $j = k+1, k+2, \ldots, n$, we have a constraint
  $$(x_i - x)^2 + (y_i - y)^2 \leqq (x_j - x)^2 + (y_j - y)^2$$
- From this inequality, we have a constraint
  $$2(x_i - x_j)x + 2(y_i - y_j)y \geqq x_i^2 + y_i^2 - x_j^2 - y_j^2$$

Formulation (2)

From Naoki Katoh, Mathematical programming, Corona publishing, p. 58【7】

- Formulated linear programming problem is as follows:

Determine whether we have a feasible solution (a, b, c) satisfying the constraints



Note that x_i, y_i, x_j, y_j are constants

- minimize 0
- subject to

$$2\,(x_i - x_j)\,x + 2\,(y_i - y_j)\,y \geqq x_i^2 + y_i^2 - x_j^2 - y_j^2$$

$$(i = 1, 2, \ldots, k;\ \ j = k+1, k+2, \ldots, n)$$

※ By solving this linear programming problem, we can obtain the center (x, y) of the circle. The radius is not obtained.

Summary

- Simplex method
  - How to find an initial feasible basic solution ?
  - How to select a pivot ?
- Situations we need to consider
  → Two-phase simplex method
- Formulation of linear programming problems