Large-Scale Knowledge Processing #7
Optimization Techniques (Part 4)

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Upper bound of the optimal value

- Max. prob. : Upper bound of the optimal value ? ($z \leqq$ ○○)

$$
\begin{array}{llr}
\textbf{maximize} & z = 2x_1 + 3x_2 + 4x_3 & \\
\text{subject to} & x_1 + 2x_2 \leqq 5 & \cdots (1) \\
& x_2 + 2x_3 \leqq 3 & \cdots (2) \\
& x_1 + x_3 \leqq 2 & \cdots (3) \\
& x_1, x_2, x_3 \geqq 0 &
\end{array}
$$

- By **2** (1) + **2** (2), we have
  - $2x_1 + 6x_2 + 4x_3 \leqq 16$
- Since $0 \leqq x_2$, we have
  - $z = 2x_1 + 3x_2 + 4x_3 \leqq 2x_1 + 6x_2 + 4x_3 \leqq 16$

Mix (1), (2)

Every **coefficient** of x_1, x_2, x_3 in the mixed **inequality** is larger than or equal to that in the **objective function**

Can we obtain an upper bound better than 16 ?

Upper bound of the optimal value

- Max. prob. : Upper bound of the optimal value ? ($z \leqq$ ○○)

$$
\begin{array}{llr}
\textbf{maximize} & z = 2x_1 + 3x_2 + 4x_3 & \\
\text{subject to} & x_1 + 2x_2 \leqq 5 & \cdots (1) \\
& x_2 + 2x_3 \leqq 3 & \cdots (2) \\
& x_1 + x_3 \leqq 2 & \cdots (3) \\
& x_1, x_2, x_3 \geqq 0 &
\end{array}
$$

Notice: To keep the sign of the inequalities, we assume $y_1, y_2, y_3 \geqq 0$

- By y_1 (1) + y_2 (2) + y_3 (3),
  - $(y_1 + y_3) x_1 + (2y_1 + y_2) x_2 + (2y_2 + y_3) x_3 \leqq 5y_1 + 3y_2 + 2y_3$
- To obtain an upper bound from **this inequality**, we compare the **coefficients** with those in the **objective function**
  - $y_1 + y_3 \geqq 2, \quad 2y_1 + y_2 \geqq 3, \quad 2y_2 + y_3 \geqq 4$
- Min. $5y_1 + 3y_2 + 2y_3$ … minimize the upper bound

Dual problem

- Primal prob. (P)

$$\begin{array}{llr} \textbf{maximize} & z = 2x_1 + 3x_2 + 4x_3 & \\ \text{subject to} & x_1 + 2x_2 & \leqq 5 \\ & x_2 + 2x_3 & \leqq 3 \\ & x_1 + x_3 & \leqq 2 \\ & x_1, x_2, x_3 \geqq 0 & \end{array}$$

$$\text{maximize} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leqq \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

$$x_1, x_2, x_3 \geqq 0$$

- Dual prob. (D): the dual of the promal prob.

$$\begin{array}{llr} \textbf{minimize} & w = 5y_1 + 3y_2 + 2y_3 & \\ \text{subject to} & y_1 + y_3 & \geqq 2 \\ & 2y_1 + y_2 & \geqq 3 \\ & 2y_2 + y_3 & \geqq 4 \\ & y_1, y_2, y_3 \geqq 0 & \end{array}$$

$$\text{minimize} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}^T \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}^T \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \geqq \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}^T$$

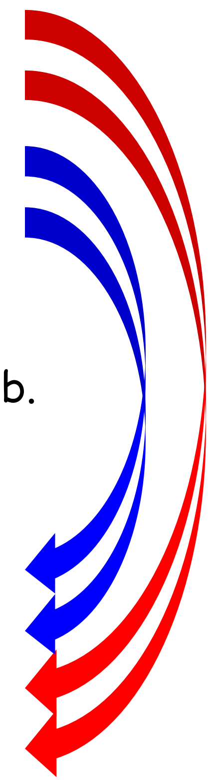
$$y_1, y_2, y_3 \geqq 0$$

- The dual of the dual prob. is the primal prob.

Dual problem

- Primal prob. (P) (General form)

minimize	$z = c^T x$	
subject to	$a_i^T x = b_i$	$(i \in M)$
	$a_i^T x \geqq b_i$	$(i \in M')$
	$x_j \geqq 0$	$(j \in N)$
	x_j free variable	$(j \in N')$



- Dual prob. (D): the dual of the promal prob.

maximize	$w = y^T b$	
subject to	$y^T A_j \leqq c_j$	$(j \in N)$
	$y^T A_j = c_j$	$(j \in N')$
	y_i^T free variable	$(i \in M)$
	$y_i^T \geqq 0$	$(i \in M')$

- The dual of the dual prob. is the primal prob.

Dual prob., Weak duality theorem

- Primal prob. (P) (Standard form)

minimize	$z = c^T x$
subject to	$A x = b$
	$x \geqq 0$

- Dual prob. (D)

maximize	$w = y^T b$
subject to	$y^T A \leqq c^T$
	y^T free variables

- The dual of the dual prob. is the primal prob.
- ***Weak duality theorem***

  $y^T b = y^T A x \leqq c^T x$

  - Let x, y be feasible solutions of (P), (D), respectively. Then, we have inequality $y^T b \leqq c^T x$

Practice: Dual prob.

Obtain the dual probs. of the following probs.

(a)

$$
\begin{aligned}
\textbf{maximize} \quad & z = 4x_1 + 3x_2 + 2x_3 \\
\text{subject to} \quad & x_1 + 2x_2 = 7 \\
& x_2 + 2x_3 \leqq 8 \\
& x_1 + x_3 \leqq 9 \\
& x_1, x_2 \geqq 0
\end{aligned}
$$

(b)

$$
\begin{aligned}
\textbf{minimize} \quad & z = 4x_1 + 3x_2 + 2x_3 \\
\text{subject to} \quad & x_1 + 2x_2 = 7 \\
& x_2 + 2x_3 \geqq 8 \\
& x_1 + x_3 \geqq 9 \\
& x_1, x_2 \geqq 0
\end{aligned}
$$

Practice: Dual prob., Weak duality theorem

1. Obtain the standard forms and its dual problems for the following problems:
   a. The problem in lecture #6 slide p. 9
   b. #6 p. 10
   c. #3 p. 29 (a)
   d. #3 p. 29 (b)
   e. #3 p. 20 (a)
   f. #3 p. 20 (c)

2. Show that the weak duality theorem holds.

Weak duality theorem

Corollary

Primal prob. (P)	**min.**	$c^T x$
Dual prob. (D)	**max.**	$y^T b$
Weak D. Thm.		$y^T b \leqq c^T x$

- From the weak duality theorem, we can derive the following corollary
- Suppose that x and y are feasible solutions of the primal prob. **(P)** and the dual prob. **(D)**, respectively. If x and y satisfy $\mathbf{c^T x = y^T b}$,

x and y are the **optimal solutions** of (P) and (D), respectively.

Corollary

Primal prob. (P) **min.** $c^T x$
Dual prob. (D) **max.** $y^T b$
Weak D. Thm. $y^T b \leqq c^T x$

- From the weak duality theorem, we can derive the following corollary

- If **(P)** is **unbounded**, **(D)** is **infeasible**
- If **(D)** is **unbounded**, **(P)** is **infeasible**

Proof)

- If (P) is unbounded, we can minimize $c^T x$ arbitrarily
- If (D) has a feasible solution y, any feasible solution x of (P) has lower bound $c^T x \geqq y^T b$
- Contradiction → (D) is infeasible

- Same argument holds for "If (D) is unbounded…"

Strong duality theorem

- If the primal prob. (P) has an **optimal solution** x^*, the dual prob. (D) also has an **optimal solution** y^*, and **the optimal values** of the probs. **are equal**
  … $c^T x^* = y^{*T} b$
- Relations between the primal and dual probs.

			(D)		
			feasible		infeasible
			opt. solution	unbounded	
(P)	feasible	opt. solution	○	×	×
		unbounded	×	×	○
	infeasible		×	○	○

We only have ○ cases (i.e., × cases do not occur)

Ex.) Both primal and dual are infeasible

- Primal prob. (P)

$$\begin{array}{ll} \text{mimimize} & z = -x_1 - x_2 \\ \text{subject to} & x_1 - x_2 = 1 \\ & x_1 - x_2 = 0 \\ & x_1, x_2 \geqq 0 \end{array}$$

- Dual prob. (D)

$$\begin{array}{ll} \text{maximize} & w = y_1 \\ \text{subject to} & y_1 + y_2 \leqq -1 \\ & -y_1 - y_2 \leqq -1 \\ & y_1, y_2 : \text{free variables} \end{array}$$

$y_1 + y_2 \geqq 1$

practice : Strong duality theorem

- We go back to the problem in lecture #6 slide p. 11. (The feasible region of this problem is unbounded.) Show its standard form and its dual, and confirm that the dual problem is infeasible.

$$
\begin{array}{ll}
\text{mimimize} & z = -x_1 - 2x_2 \\
\text{subject to} & x_1 + x_2 \geqq 1 \\
 & x_1, x_2 \geqq 0
\end{array}
$$

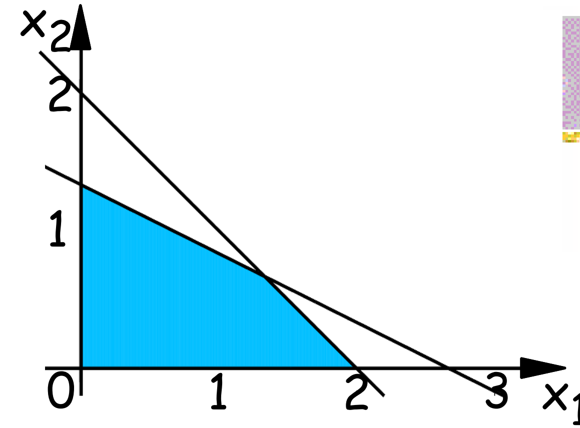
practice :

For each of the problems below, show its standard forms and its dual, and check whether the primal and dual problems have optimal solutions, is unbounded, or infeasible.

a. The problem in lecture #6 slide p. 9

b. #6 p. 10

Methods for linear programming



- Simplex method
  - obtains an optimal solution
    by traversing the boundary of the feasible region
    (by walking from an extreme point to another extreme point)
- Internal point method
  - obtains an optimal solution
    by walking through the interior of the feasible region

Speed-up

[Bixby 2002] Solving real-world linear programs: a decade and more of progress

- by hardware: × 800 faster
- by algorithm: × 2,400 faster

} × 1,900,000

[Bertsimas, King, Mazumder 2016]
According to tireless effort, we have × 450,000,000 speed-up
in about 25 years

Materials

- 加藤直樹, 数理計画法, コロナ社
  ISBN 978-4339027198
- 宮代隆平, 整数計画ソルバー入門,
  オペレーションズ・リサーチ, 57-4 (2012), pp. 183-189.
- Recommended books, blogs, and more (Grobi)
  https://www.gurobi.com/resources/books-blog/

Solver

- Proprietary software:
  Gurobi, CPLEX, Optimization Toolbox (Matlab), solver for Excel …
- Free software:
  SCIP, MIPCL, GLPK, lp_solve …