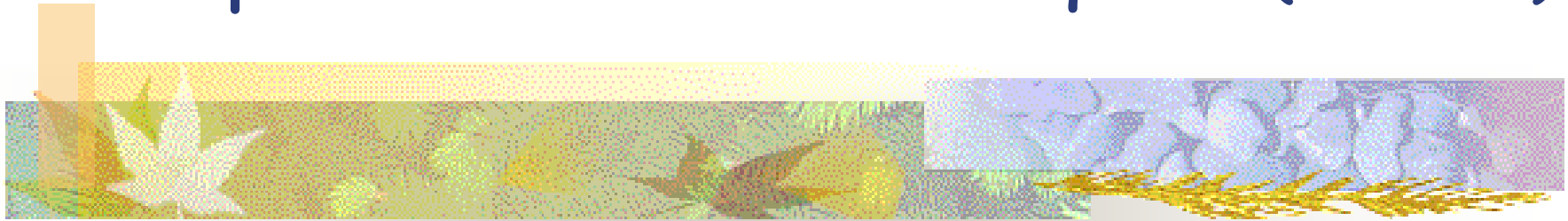


Large-Scale Knowledge Processing Optimization Techniques (Part 1)



Faculty of Information Science
and Technology, Hokkaido Univ.

Takashi Horiyama

Mathematical programming and optimization

Mathematical programming model



- Find the optimal solution
- What is the target for the optimization?
- Objective of the optimization?

Production planning problem

- Maximum profit ?

We produce two products 1 and 2
from two materials A and B

- Material usage per unit of product
and the amount of available materials

	Product 1 (10^3 kg)	Product 2 (10^3 kg)	Available materials (10^3 kg)
Material A	2	2	4
Material B	3	6	8

- Profit per unit of product

- Product 1: 4

- Product 2: 5

Formulation of production planning

- Represent as mathematical models
- Let x_1, x_2 be the volumes of products 1, 2
- Objective: Maximize the profit
 - Maximize $4x_1 + 5x_2$
- Condition: Available materials
 - Material A: $2x_1 + 2x_2 \leq 4$
 - Material B: $3x_1 + 6x_2 \leq 8$
- Condition: Volumes of the products should be non-negative
 - Product 1: $x_1 \geq 0$
 - Product 2: $x_2 \geq 0$

Formulation of production planning

- Let x_1, x_2 be the volumes of products 1, 2
← decision variables, **variables**
- maximize $4 x_1 + 5 x_2$ ← **Objective function**
- subject to $2 x_1 + 2 x_2 \leq 4$ ← **Constraints**
 $3 x_1 + 6 x_2 \leq 8$
 $x_1 \geq 0, x_2 \geq 0$
- **Linear programming problem**
 - Objective function is a **linear** function
 - maximize or minimize the objective function
 - All constraints are **linear** inequalities or equalities

Production planning problem

- If the volumes of the products are limited to **integers** ...

- maximize $4 x_1 + 5 x_2$

subject to $2 x_1 + 2 x_2 \leq 4$

$$3 x_1 + 6 x_2 \leq 8$$

$$x_1 \geq 0, x_2 \geq 0, \quad x_1, x_2 \text{ are integers}$$

- **Integer programming problem**

- Variables are limited to be integers
- 0–1 integer programming problem
 - ... Variables are 0 or 1
- Mixed integer programming problem
 - ... Some variables are integers/Others are real numbers

practice: Formulation of product planning

- Maximum profit ?

We produce three products 1, 2 and 3 from four materials A, B, C and D

- Material usage per unit of product and the amount of available materials

	Product 1 (10 ³ kg)	Product 2 (10 ³ kg)	Product 3 (10 ³ kg)	Available Material (10 ³ kg)
Material A	4	2	1	6
Material B	1	2	4	7
Material C	5	2	3	9
Material D	3	3	2	8

- Profit per unit of product

- Product 1: 3, Product 2: 5, Product 3: 4

practice: Formulation of product planning

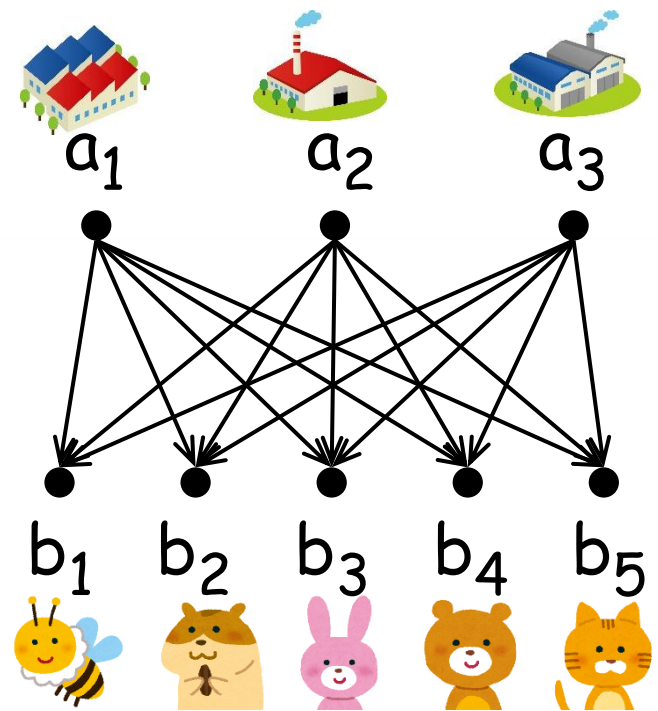
■ Maximum profit ?

We produce n kinds of products P_1, P_2, \dots, P_n
from m kinds of materials S_1, S_2, \dots, S_m

- c_{ij} : Necessary amount of material S_j
for producing a unit of product P_i
- b_j : Available amount of material S_j
- a_i : Profit per unit of product P_i

Transportation problem

- Transport products from factories 1, 2, 3 to consumers 1, 2, 3, 4, 5
- Minimize transportation costs



- a_i : The amount of products factory i can ship in a month
- b_j : Demand by consumer j in a month
- c_{ij} : Transportation cost (per unit of product) from factory i to consumer j

Transportation problem

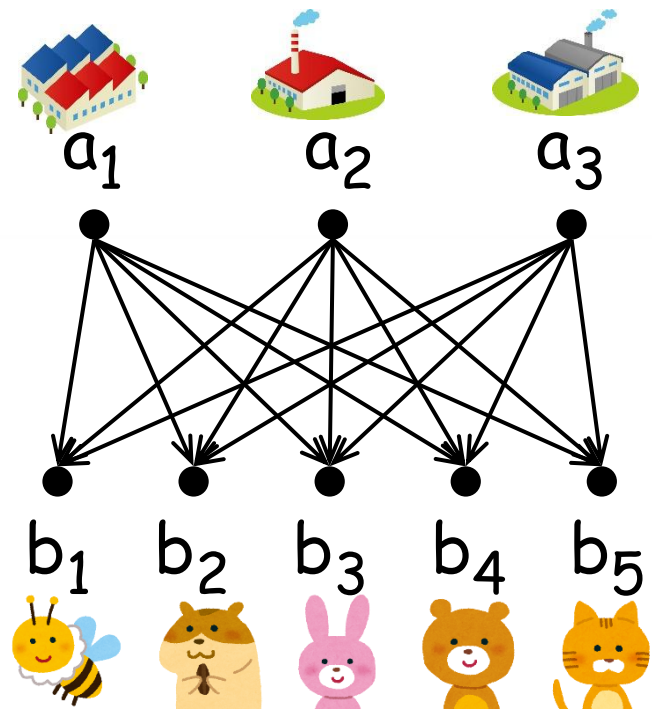
- Let x_{ij} be the amount of the products shipped from factory i to consumer j

- minimize
$$\sum_{i=1}^3 \sum_{j=1}^5 c_{ij} x_{ij}$$

subject to
$$\sum_{j=1}^5 x_{ij} \leq a_i \quad (i = 1, 2, 3)$$

$$\sum_{i=1}^3 x_{ij} \geq b_j \quad (j = 1, 2, \dots, 5)$$

$$x_{ij} \geq 0 \quad (i = 1, 2, 3, j = 1, 2, \dots, 5)$$

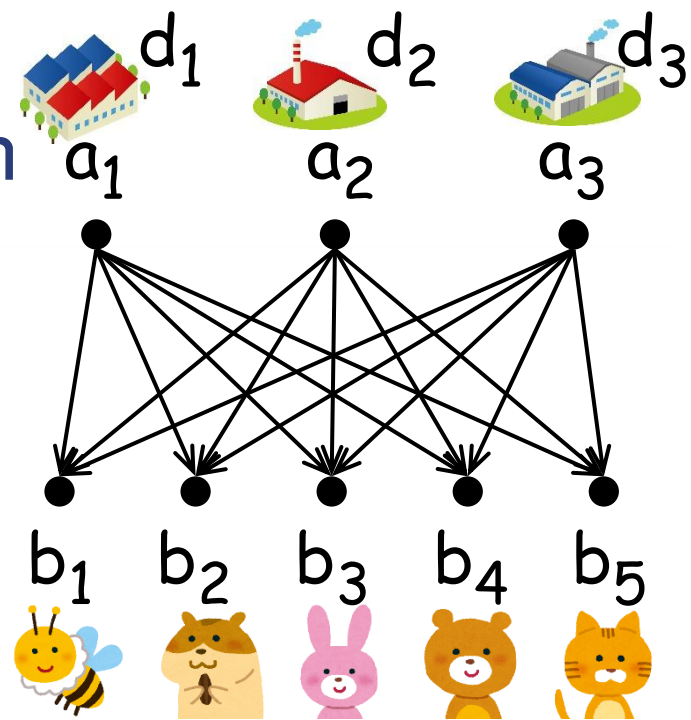


Factory i

Consumer j

Facility location problem

- In the problem in p. 9 we need cost d_i for running factory i
(In case we do not use factory i , its cost becomes 0)
- We need to minimize the sum of the transportation cost and the running cost of the factories



Formulation of facility location problem

- x_{ij} : the amount from factory i to consumer j
 y_i : run factory i , or not
($y_i = 1 \cdots$ run factory i , $y_i = 0 \cdots$ do not use factory i)

- minimize
$$\sum_{i=1}^3 \sum_{j=1}^5 c_{ij} x_{ij} + \sum_{i=1}^3 d_i y_i$$

Factory i

$$\sum_{j=1}^5 x_{ij} \leq a_i y_i$$

$$\left\{ \begin{array}{ll} \sum_{j=1}^5 x_{ij} \leq a_i & \text{Run factory } i \ (y_i = 1) \\ \sum_{j=1}^5 x_{ij} = 0 & \text{Do not use factory } i \ (y_i = 0) \end{array} \right.$$

Formulation of facility location problem

- x_{ij} : the amount from factory i to consumer j
 y_i : run factory i , or not
($y_i = 1 \cdots$ run factory i , $y_i = 0 \cdots$ do not use factory i)

- minimize
$$\sum_{i=1}^3 \sum_{j=1}^5 c_{ij} x_{ij} + \sum_{i=1}^3 d_i y_i$$

subject to
$$\sum_{j=1}^5 x_{ij} \leq a_i y_i \quad (i = 1, 2, 3)$$

Factory i

$$\sum_{i=1}^3 x_{ij} \geq b_j \quad (j = 1, 2, \dots, 5)$$

Consumer j

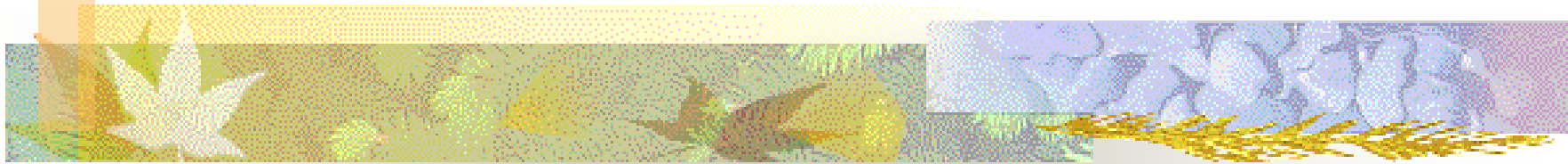
$$x_{ij} \geq 0 \quad (i = 1, 2, 3, j = 1, 2, \dots, 5)$$

$$y_i \in \{0, 1\} \quad (i = 1, 2, 3)$$

Short break

- Take a deep breath, and relax yourself

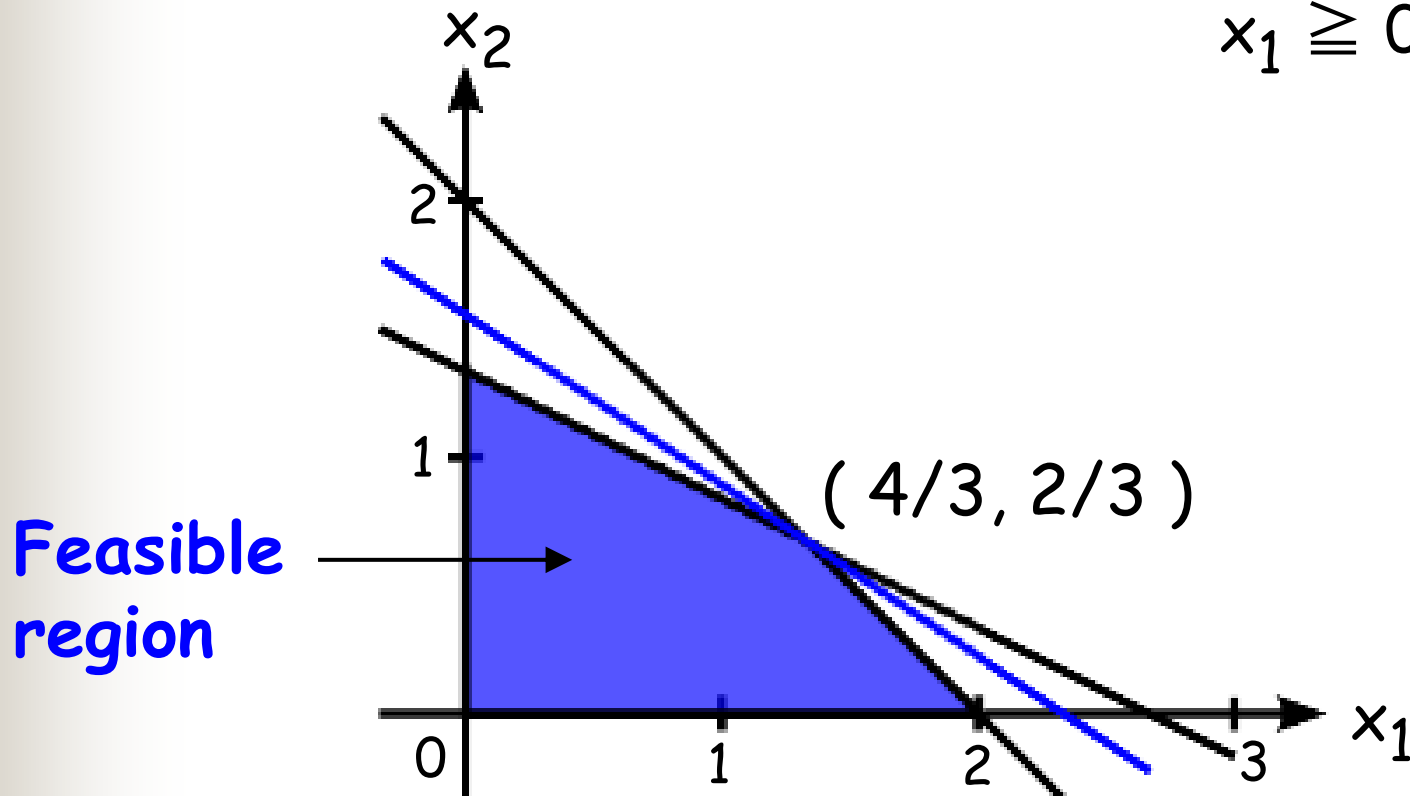
Linear programming



- How to solve linear programming problem ?

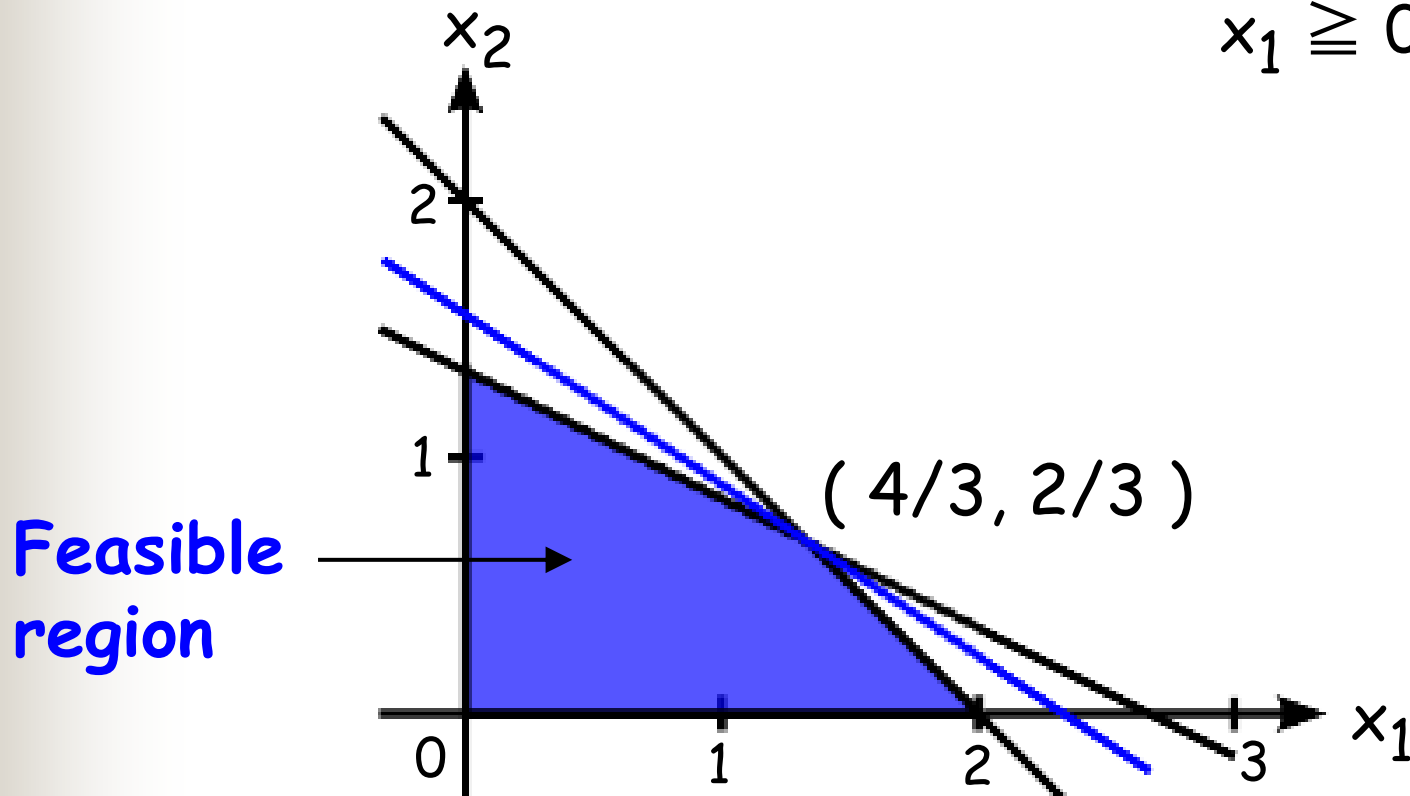
Practice: Solve by drawing

- Optimal solution ?
- maximize $4x_1 + 5x_2$
- $(x_1, x_2) =$
- subject to $2x_1 + 2x_2 \leq 4$
- Optimal value ? $3x_1 + 6x_2 \leq 8$
- $x_1 \geq 0, x_2 \geq 0$



Practice: Solve by drawing

- Optimal solution ? ■ maximize $4x_1 + 5x_2$
 $(x_1, x_2) = (4/3, 2/3)$ subject to $2x_1 + 2x_2 \leq 4$
- Optimal value ? $26/3$
 $3x_1 + 6x_2 \leq 8$
 $x_1 \geq 0, x_2 \geq 0$



Practice: Solve by drawing (Integer programming)

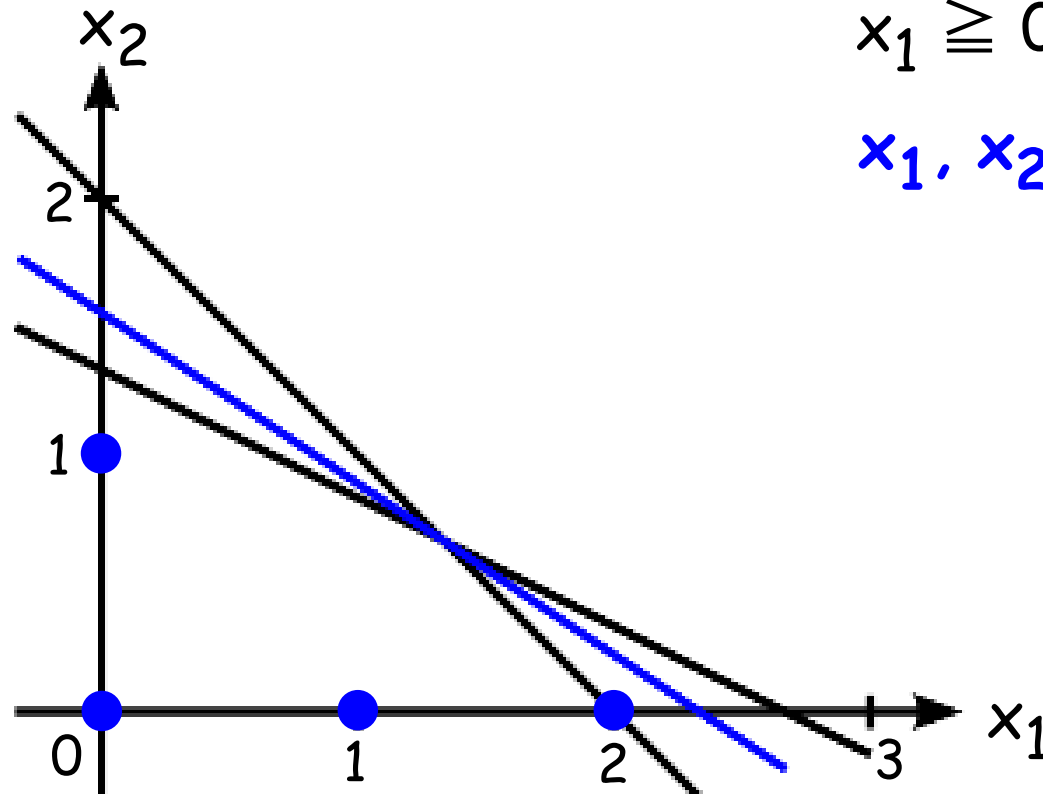
- Optimal solution ?
 $(x_1, x_2) =$
- Optimal value ?

■ maximize $4x_1 + 5x_2$
subject to $2x_1 + 2x_2 \leq 4$

$$3x_1 + 6x_2 \leq 8$$

$$x_1 \geq 0, x_2 \geq 0$$

x_1, x_2 are integers



Feasible set

Practice: Solve by drawing (Integer programming)

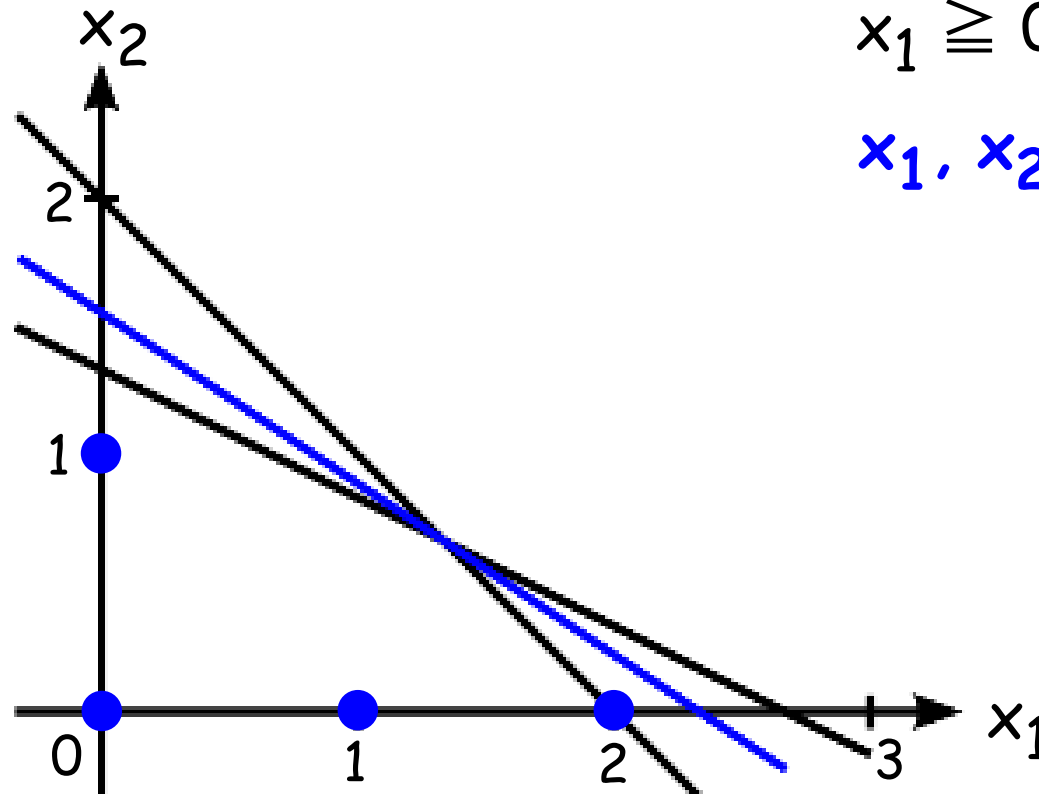
- Optimal solution ?
 $(x_1, x_2) = (2, 0)$
- Optimal value ? 8

■ maximize $4x_1 + 5x_2$
subject to $2x_1 + 2x_2 \leq 4$

$$3x_1 + 6x_2 \leq 8$$

$$x_1 \geq 0, x_2 \geq 0$$

x_1, x_2 are integers



Feasible set

Practice: Solve by drawing

(a) maximize $4x_1 + 5x_2$
subject to $2x_1 + 2x_2 \leq 4$
 $3x_1 + 6x_2 \leq 8$
 $x_1 + 4x_2 \leq 4$
 $x_1 \geq 0, x_2 \geq 0$

(b) Change prob. (a)
to minimization
problem

(c) maximize $4x_1 + 2x_2$
subject to $x_1 + x_2 \leq 8$
 $x_1 \leq 6$
 $x_1 + 2x_2 \geq 2$
 $x_1 \geq 0, x_2 \geq 0$

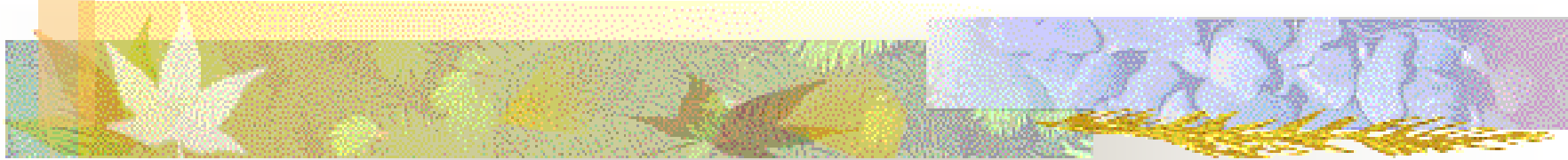
(d) Change prob.(c)
to minimize
problem

Advanced: Solve by drawing (let's challenge)

- Solve the linear programming problem in slide 7
 - **3-dimensional space**
 - $4x_1 + 2x_2 + x_3 \leq 6$
... half space cut by a plane
- It is very difficult to solve linear programming problems with 3 (or more) variables by drawing ...

Linear programming

Standard form



Standard form

- We can translate all linear programming problems into their **standard forms**

- **minimize** $z = \sum_{j=1}^n c_j x_j$

subject to $\sum_{j=1}^n a_{ij} x_j = b_i \quad (i = 1, 2, \dots, m)$

$$x_j \geq 0 \quad (j = 1, 2, \dots, n)$$

ただし、 **$b_i \geq 0$** (※ Do not forget this)

Ex.) Transformation

- maximize $z = 2x_1 - 3x_2 - 4x_3$
subject to $2x_1 + x_2 - 3x_3 \leq 4$
 $x_1 - x_2 + 4x_3 \geq 5$
 $x_1, x_2 \geq 0, x_3$: free variable

- maximize $z = 2x_1 - 3x_2 - 4x_3$
- minimize $z = -2x_1 + 3x_2 + 4x_3$

Use “minimize”

Ex.) Transformation (cont.)

- maximize $z = 2x_1 - 3x_2 - 4x_3$
subject to $2x_1 + x_2 - 3x_3 \leq 4$
 $x_1 - x_2 + 4x_3 \geq 5$
 $x_1, x_2 \geq 0, x_3$: free variable

- $2x_1 + x_2 - 3x_3 \leq 4$
- $2x_1 + x_2 - \underline{3x_3} + \mathbf{s_1} = 4, s_1 \geq 0$

introduce a
slack variable

Ex.) Transformation (cont.)

- maximize $z = 2x_1 - 3x_2 - 4x_3$
subject to $2x_1 + x_2 - 3x_3 \leq 4$
 $x_1 - x_2 + 4x_3 \geq 5$
 $x_1, x_2 \geq 0, x_3$: free variable

Both variables are called **slack variables** in many cases

- $2x_1 + x_2 - 3x_3 \leq 4$
- $2x_1 + x_2 - 3x_3 + s_1 = 4, s_1 \geq 0$

introduce a **slack variable**

- $x_1 - x_2 + 4x_3 \geq 5$
- $x_1 - x_2 + 4x_3 - s_2 = 5, s_2 \geq 0$

introduce a **surplus variable**

Ex.) Transformation (cont.)

- maximize $z = 2x_1 - 3x_2 - 4x_3$
 subject to $2x_1 + x_2 - 3x_3 \leq 4$
 $x_1 - x_2 + 4x_3 \geq 5$
 $x_1, x_2 \geq 0, x_3$: free variable

Both variables are called **slack variables** in many cases

introduce a **slack variable**

introduce a **surplus variable**

introduce **non-negative** variables

- $2x_1 + x_2 - 3x_3 \leq 4$
- $2x_1 + x_2 - 3x_3 + s_1 = 4, s_1 \geq 0$
- $x_1 - x_2 + 4x_3 \geq 5$
- $x_1 - x_2 + 4x_3 - s_2 = 5, s_2 \geq 0$
- $2x_1 + x_2 - 3x_3' + 3x_3'' + s_1 = 4$
- $x_1 - x_2 + 4x_3' - 4x_3'' - s_2 = 5, x_3', x_3'' \geq 0$

Ex.) Transformation (summary)

■ maximize $z = 2x_1 - 3x_2 - 4x_3$
subject to $2x_1 + x_2 - 3x_3 \leq 4$
 $x_1 - x_2 + 4x_3 \geq 5$
 $x_1, x_2 \geq 0, x_3$: free variable

■ minimize $z = -2x_1 + 3x_2 + 4x_3' - 4x_3''$
subject to $2x_1 + x_2 - 3x_3' + 3x_3'' + s_1 = 4$
 $x_1 - x_2 + 4x_3' - 4x_3'' - s_2 = 5$
 $x_1, x_2, x_3', x_3'', s_1, s_2 \geq 0$

practice: Standard form

(a) maximize $z = 4x_1 + 2x_2$
subject to $2x_1 + 2x_2 \leq 4$
 $-3x_1 - 6x_2 \leq -9$
 $x_1 \geq 0, x_2$: free variable

(b) maximize $z = 3x_1 + 5x_2 + 4x_3$
subject to $4x_1 + 2x_2 + 3x_3 \leq 6$
 $3x_1 - 4x_2 + 5x_3 \leq -2$
 $x_1 \geq 0, x_2 \geq 0, x_3$: free variable

Summary

- Formulation of the problems
 - **Linear programming problem**
 - **Integer programming problem**
- Solve linear programming problems
 - Solve by drawing
 - It is very difficult to solve linear programming problems with 3 (or more) variables
- **Standard form** of a linear programming problem
 - Any linear programming problem can be transformed into its standard form



