Large-Scale Knowledge Processing

Optimization Techniques (2)

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Prev. class + α : Linear Programming Problem Standard form

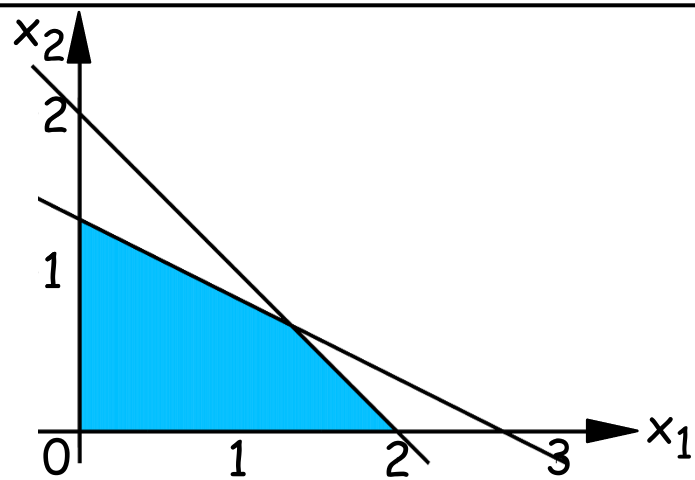
- minimize $z = -4x_1 - 5x_2$
  subject to
  $$2x_1 + 2x_2 + x_3 = 4$$
  $$3x_1 + 6x_2 + x_4 = 8$$
  $$x_1, x_2, x_3, x_4 \geqq 0$$

Transformation to its standard form

maximize $z = 4x_1 + 5x_2$
subject to
$$2x_1 + 2x_2 \leqq 4$$
$$3x_1 + 6x_2 \leqq 8$$
$$x_1, x_2 \geqq 0$$

Intuitive observation :

- By checking the **extreme points** (vertices) of the feasible region, we can find the optimal solution
- How can we find **extreme points** of the feasible region ? (The standard form gives a good hint)



Ex.) Standard form (Vector representation)

- minimize $z = -4x_1 - 5x_2$

  subject to
  $$2x_1 + 2x_2 + x_3 = 4$$
  $$3x_1 + 6x_2 + 4x_4 = 8$$
  $$x_j \geqq 0 \quad (j = 1, 2, \dots, 4)$$

- minimize
  $$z = \begin{pmatrix} -4 & -5 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

subject to
$$\begin{pmatrix} 2 & 2 & 1 & 0 \\ 3 & 6 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geqq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Standard form (Vector representation)

$$A = \begin{pmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$$

- minimize $z = \sum_{j=1}^{n} c_j x_j$

  subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i \quad (i = 1, 2, \ldots, m)$

  $x_j \geqq 0 \quad (j = 1, 2, \ldots, n)$

- minimize $z = c^T x$

  subject to $A x = b$

  $x \geqq 0$

$c^T = (c_1, c_2, \ldots, c_n)$

$A = (a_{ij})$ $m \times n$ matrix

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Standard form (Vector representation)

$$A = \begin{pmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

A can be regarded as $(A_1 \ A_2 \cdots A_n)$

(i.e., concatenation of n vertical vectors)

- minimize $z = c^T x$

  subject to $A x = b$

  $x \geqq 0$

$c^T = (c_1, c_2, \ldots, c_n)$

$A = (a_{ij})$ $m \times n$ matrix

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Ex.) Basis

Constraints in p. 3

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 3 & 6 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

- Infinitely many assignments to (x_1, x_2, x_3, x_4) satisfying the 2 conditions
  - In case $(x_3, x_4) = (0, 0)$?
  - i.e., if we **focus on x_1, x_2**

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 3 & 6 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

→ (x_1, x_2, x_3, x_4) is uniquely obtained

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 3 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 2/3 \end{pmatrix}$$

  - **Basic vectors**: $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix} \right\}$
  - **Basic matrix**: $\begin{pmatrix} 2 & 2 \\ 3 & 6 \end{pmatrix}$
  - **Basic variables**: x_1, x_2, **Nonbasic variables**: x_3, x_4
  - **Basic solution** for basic vectors $\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix} \right\}$: $\hat{x} = (4/3, 2/3, 0, 0)$

Ex.) Basis

Constraints in p. 3

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 3 & 6 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

- Try another base vector

$$\begin{bmatrix} 2 & 2 & 1 & 0 \\ 3 & 6 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 8/3 \\ -4/3 \end{pmatrix}$$

- **Basic vectors**: $\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}$
- **Basic matrix**: $\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$
- **Basic variables**: x_1, x_3, **Nonbasic variables**: x_2, x_4
- **Basic solution** for basic vectors $\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}$: $\hat{x} = (8/3, 0, -4/3, 0)$

Supplementary information

- Some basic matrix **may not have the inverse matrix**

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$ are linear dependent (not independent)

- $A = \begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$
  - Assumption 1: $n \geqq m$
  - Assumption 2: the rank of A is m
  - Assumption 3: optimal solution exists

We can relax these assumptions (later)

Basic vector
Basic matrix

$$A = \begin{pmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} & \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} & \cdots & \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \end{pmatrix} = (A_1 \quad A_2 \quad \cdots \quad A_n)$$

Their combination is not unique

- **Basic vector**
  - **m** linear independent vectors in A
    Basis $\mathcal{B} = \{ A_{i_1}, A_{i_2}, \ldots, A_{i_m} \}$
- **Non basic vector**
  - Other **(n - m)** vectors

$$\begin{pmatrix} 2 & 2 & 1 & 0 \\ 3 & 6 & 0 & 4 \end{pmatrix}$$

$$\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix} \right\}$$

- **Basic matrix** B
  - m x m square matrix with **m** basic vectors
    $B = (A_{i_1} \quad A_{i_2} \cdots A_{i_m})$
- **Nonbasic matrix** N
  - m x (n - m) matrix with **(n - m)** Nonbasic vectors

$$\begin{pmatrix} 2 & 2 \\ 3 & 6 \end{pmatrix}$$

Basic variable
Basic solution

$x_B = (x_1, x_2)$

$x_N = (x_3, x_4)$

$\hat{x} = (8/3, 2/3, 0, 0)$

- **Basic variable**
  - **m** basic variables $x_{i_1}, x_{i_2}, \ldots, x_{i_m}$ corresponding to m basic vectors $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$
  - m-dimensional vector $x_B = (x_{i_1}, x_{i_2}, \ldots, x_{i_m})$
- **Nonbasic variable**
  - Other **(n−m)** variables
  - (n−m)-dimensional vector x_N
- **Basic solution $\hat{x}$** for basis $\mathcal{B} = \{A_{i_1}, A_{i_2}, \ldots, A_{i_m}\}$
  - $$x_j = \begin{cases} 0 & (A_j \notin \mathcal{B}) \\ \ell\text{-th element of } B^{-1}b & (A_j \in \mathcal{B}, x_j = x_{i_\ell}) \end{cases}$$

Solution of $Bx = b$

Standard form in p. 3

practice : Basic solution

- minimize $z = -4x_1 - 5x_2$

  subject to $2x_1 + 2x_2 + x_3 = 4$

  $3x_1 + 6x_2 + x_4 = 8$

  $x_j \geqq 0 \ (j = 1, 2, \ldots, 4)$

- $A_1 = \binom{2}{3}$, $A_2 = \binom{2}{6}$, $A_3 = (\quad)$, $A_4 = (\quad)$
- A_1, A_2 independent : $\mathrm{rank}(A_1 A_2) = m$ (determinant≠0)
- Solve $(A_1 \ A_2)\binom{4}{8} = \binom{x_1}{x_2}$: we have $x_1 =$, $x_2 =$
- Basic solution for A_1, A_2 is $x^T = (\ , \ , \ , \ )$
- All basic elements (elements of basic solution) are non-negative → **basic feasible solution**

practice： Basic solution

- From the conditions of the problem in p. 3, we have 4 vectors $A_1 \sim A_4$. We have 6 ways for choosing 2 basic vectors A_i and A_j. For each of them, find basic solutions.
- From each of the above basic solutions, obtain the coordinate (x_1, x_2). Find such coordinates in the graph of the original optimization problem.

Original optimization prob.

$$
\begin{aligned}
\text{maximize} \quad & 4x_1 + 5x_2 \\
\text{subject to} \quad & 2x_1 + 2x_2 \leqq 4 \\
& 3x_1 + 6x_2 \leqq 8 \\
& x_1 \geqq 0,\ x_2 \geqq 0
\end{aligned}
$$

Standard form in p. 3

Standard form, basic solution

$Ax = b$
$x \geqq 0$

- minimize $z = -4x_1 - 5x_2$
  subject to
  $2x_1 + 2x_2 + x_3 = 4$
  $3x_1 + 6x_2 + x_4 = 8$
  $x_1, x_2, x_3, x_4 \geqq 0$

$A_1 \; A_2 \; A_3 \; A_4$

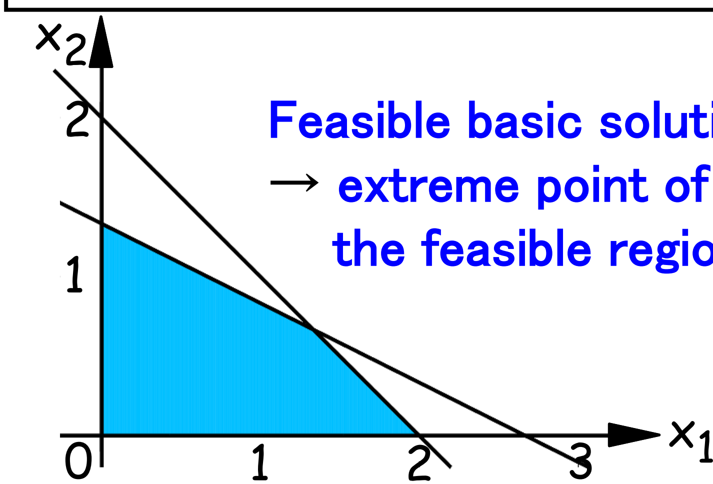
$\begin{pmatrix}2\\3\end{pmatrix} \begin{pmatrix}2\\6\end{pmatrix} \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix}$

Transformation to its standard form

maximize $z = 4x_1 + 5x_2$
subject to
$2x_1 + 2x_2 \leqq 4$
$3x_1 + 6x_2 \leqq 8$
$x_1, x_2 \geqq 0$

basic solutions for

- A_1, A_2: $(\frac{4}{3}, \frac{2}{3}, 0, 0)$
- A_1, A_3: $(\frac{8}{3}, 0, -\frac{4}{3}, 0)$
- A_1, A_4: $(2, 0, 0, 2)$
- A_2, A_3: $(0, \frac{4}{3}, \frac{4}{3}, 0)$
- A_2, A_4: $(0, 2, 0, -4)$
- A_3, A_4: $(0, 0, 4, 8)$



Feasible basic solution → extreme point of the feasible region

Degeneration of feasible basic solutions

- Feasible basic solution x is **degenerate**
  - x has more 0's than n−m
- Two different basis correspond to the same basic solution x

  → x is degenerate

practice : Degeneration

- For the problems in "Optimization Techniques (1)" slides p. 20 (a), p. 21

  (1) Transform them into their standard forms

  (2) Check whether they have degenerate feasible basic solution

Degeneration of feasible basic solutions

- minimize $z = -4x_1 - 5x_2$
  subject to
  $2x_1 + 2x_2 + x_3 = 4$
  $3x_1 + 6x_2 + x_4 = 8$
  $x_1 + 4x_2 + x_5 = 4$
  $x_1, x_2, x_3, x_4, x_5 \geqq 0$

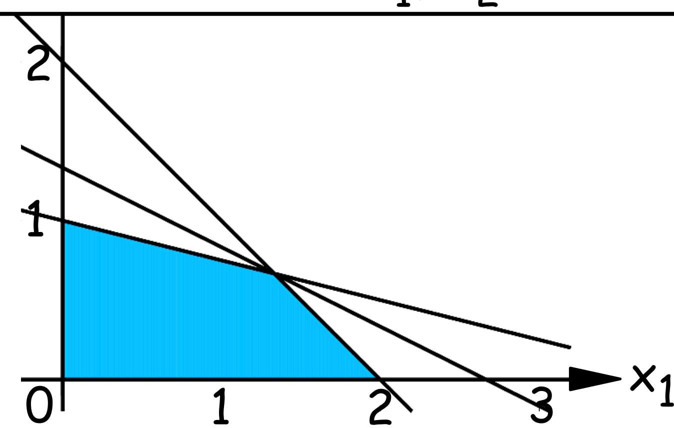
$$A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5$$
$$\begin{pmatrix}2\\3\\1\end{pmatrix}\begin{pmatrix}2\\6\\4\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

maximize $z = 4x_1 + 5x_2$
subject to
$2x_1 + 2x_2 \leqq 4$
$3x_1 + 6x_2 \leqq 8$
$x_1 + 4x_2 \leqq 4$
$x_1, x_2 \geqq 0$

basic solutions for

- A_1, A_2, A_3 $\left(\frac{4}{3}, \frac{2}{3}, 0, 0, 0\right)$
- A_1, A_2, A_4 $\left(\frac{4}{3}, \frac{2}{3}, 0, 0, 0\right)$
- A_1, A_2, A_5 $\left(\frac{4}{3}, \frac{2}{3}, 0, 0, 0\right)$

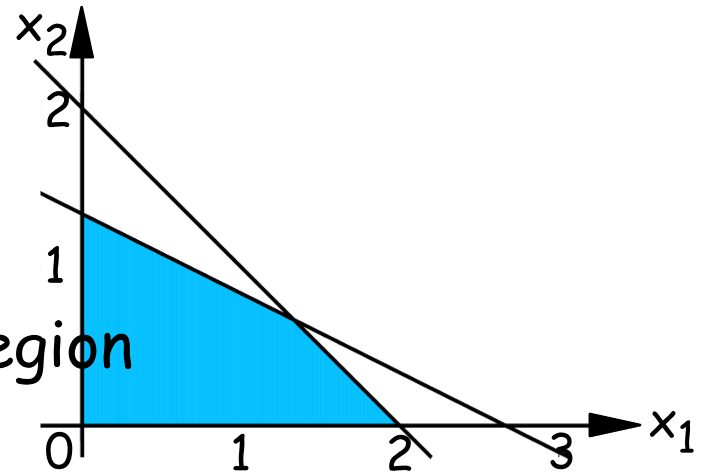
⋮



practice: Basic solution

- Enumerate all basic solutions
for the problem in p. 15
  - We have 5 basic vectors A_1, A_2, A_3, A_4, A_5
  - Choose any 3 from these 5

Summary (1st half)



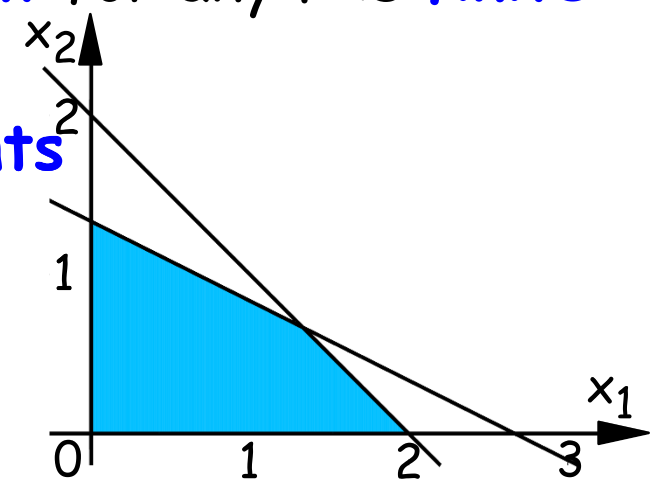
- Feasible basic solution
→ extreme point of feasible region
- Basis → basic solution
- We can find the optimal solution
by checking **all** basic solutions for **all** basis

Linear programming
Simplex method (Simplex algorithm)

- Check all basic solutions ?
- Better ways ?
  Walk through feasible basic solutions
  with improving the objective value

preliminaries: Feasible region, extreme point

- Lemma: Feasible region F is convex
- Definition: $x \in F$ is an **extreme point**
  - The point that cannot be a convex combination of two different points $x', x'' \in F$
- Properties:
  - The number of **extreme point** for any F is **finite**
  - **F** can be represented as a **convex hull** of **extreme points**



- S is convex
  - $\forall\ x', x'' \in S,\ \forall\ \alpha\ (0 \leqq \alpha \leqq 1)$
    $x' + (1-\alpha)x'' \in S$
- Convex hull of point set S'
  - Minimum convex region containing S'

Simplex dictionary (Dictionary)

- minimize $z = -2x_1 + x_2 + x_3 - x_4$

  subject to

$$x_1 + 2x_2 - x_3 + x_4 = 0$$
$$2x_1 - 2x_2 + 3x_3 + 4x_4 = 9$$
$$x_1, x_2, x_3, x_4 \geqq 0$$

- Basic solution for A_2 A_3

$\left(0, \frac{9}{4}, \frac{9}{2}, 0\right)$ ⋯ feasible solution

- Dictionary
  - Solve the conditions for **basic variables**

    + substitute them in the **objective function**

Current objective value

$$z = 27/4 - 19/4\, x_1 - 21/4\, x_4$$
$$x_2 = 9/4 - 5/4\, x_1 - 7/4\, x_4$$
$$x_3 = 9/2 - 3/2\, x_1 - 5/2\, x_4$$

Basic variables

Values in basic solution

Nonbasic variables

Basis exchange (Pivot operation)

- $z = 27/4 - 19/4\, x_1 - 21/4\, x_4$
- $x_2 = 9/4 - 5/4\, x_1 - 7/4\, x_4$
- $x_3 = 9/2 - 3/2\, x_1 - 5/2\, x_4$

- $x_1 = x_4 = 0$ holds in the current basic solution
- The coefficients of x_1, x_4 in the objective function are - 19/4, - 21/4, respectively
- If we increase x_4 by Δ
  - the objective value decreases: $27/4 - 21/4\, \Delta$
  - If x_4 increases by itself (other variables are fixed), the constraints are not satisfied

    → **change basic variables** (x_2, x_3)
    with fixing other nonbasic variables ($x_1 = 0$)

Basis exchange (Pivot operation)

- $z = 27/4 - 21/4\, x_4$
- $x_2 = 9/4 - 7/4\, x_4$
- $x_3 = 9/2 - 5/2\, x_4$

- **Fix other nonbasic variables** ($x_1 = 0$)
- Increase x_4 from 0
  - x_2 decreases (when $x_4 = 9/7$, we have $x_2 = 0$)
  - x_3 decreases (when $x_4 = 9/5$, we have $x_3 = 0$)

- Stop at $x_4 = 9/7$ to keep the **feasibility** ($x_i \geqq 0$)
  - $x_2 = 0$, i.e., "x_2 becomes a nonbasic variable"
  - x_4 becomes a basic variable

Basis exchange (Pivot operation)

- $z = 27/4 - 19/4\,x_1 - 21/4\,x_4$
- $x_2 = 9/4 - 5/4\,x_1 - 7/4\,x_4$
- $x_3 = 9/2 - 3/2\,x_1 - 5/2\,x_4$

Update a dictionary

- x_2 becomes a nonbasic variable,
  x_4 becomes a basic variable
- Transform eq. $x_2 = \ldots$ as eq. $x_4 = \ldots$
  & Substitute x_4 in other equations

- $z = -9/4 + 19/5\,x_1 + 7/5\,x_2$
- $x_4 = 9/7 - 5/7\,x_1 - 4/7\,x_2$
- $x_3 = 9/5 + 6/5\,x_1 - 2/5\,x_2$

practice : Pivot operation

- Update the following simplex dictionary so that x_2, x_3 are basic variables and x_1, x_4 are nonbasic variables

$$
\begin{aligned}
z &= \quad - 4 x_1 - 6 x_2 \\
x_3 &= 4 - 2 x_1 - 2 x_2 \\
x_4 &= 9 - 3 x_1 - 6 x_2
\end{aligned}
$$

Optimal dictionary

- All coefficients (of nonbasic variables) in the objective function are positive
  - We cannot decrease objective value

- **For all feasible solution,** $x_2, x_4 \geqq 0$ implies $z \geqq -9/4$
- $x^T = (9/5, 0, 9/5, 0)$ is **optimal** with $z = -9/4$

Optimal dictionary

Optimal value

Optimal solution $x^T = (9/5, 0, 9/5, 0)$

- $z = -9/4 + 19/5\, x_2 + 7/5\, x_4$
- $x_1 = 9/5 - 4/5\, x_2 - 7/5\, x_4$
- $x_3 = 9/5 + 6/5\, x_2 - 2/5\, x_4$

Simplex tableau

$$
\begin{aligned}
z &= 27/4 - 19/4\,x_1 - 21/4\,x_4\\
x_2 &= 9/4 - 5/4\,x_1 - 7/4\,x_4\\
x_3 &= 9/2 - 3/2\,x_1 - 5/2\,x_4
\end{aligned}
$$

- Represent a dictionary by a **simplex tableau**

		x_1	x_4
z	27/4	- 19/4	- 21/4
x_2	9/4	- 5/4	- 7/4
x_3	9/2	- 3/2	- 5/2

Current objective function: 27/4

Nonbasic variables: x_1, x_4

Basic variables: x_2, x_3

Values in basic solution: 9/4, 9/2

Relative cost coefficients: - 19/4, - 21/4

Rates of change of the objective value with respect to the nonbasic variables

- **Pivot operation**
  - Pivot (r, s) : r-th row & s-th column
  - Exchange the r-th basic var. & the s-th nonbasic var.
  - Ex.: Pivot (1, 1) to the above tableau
    - Transform eq. x_2 = … as eq. x_1 = … & Substitute x_1 in other equations

Pivot in variable (Variable entering the basis): x_1

Pivot out variable (Variable exiting from the basis): x_2

Simplex method

		x_1	x_2
z	0	- 4	- 6
x_3	4	- 2	- 2
x_4	9	- 3	- 6

minimize $z = -4\,x_1 - 6\,x_2$

subject to
$$2\,x_1 + 2\,x_2 + x_3 = 4$$
$$3\,x_1 + 6\,x_2 + x_4 = 9$$
$$x_1, x_2, x_3, x_2 \geqq 0$$

Pivot (2, 2)

Pivot the 2nd row and 2nd column

		x_1	x_4
z	-9	- 1	1
x_3	1	- 1	1/3
x_2	3/2	- 1/2	- 1/6

Pivot (,)

z		

Summary (2nd half)

- Represent a feasible basic solution (an extreme point of the feasible region) by a simplex tableau
- Overview of Simplex method
  - Find a starting point:
    Find a feasible basic solution (given in the next class)
  - Pivot operation:
    Exchange the r-th basic var. & the s-th nonbasic var.
    → Walk through the extreme points
    (until we cannot improve the objective function)