Large-Scale Knowledge Processing

Optimization Techniques (3)

Supplementary Material:

The Two-Phase Method

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Two-Phase Method

- When solving the original problem directly, we may encounter the following situations:
  - **Infeasible** (no feasible solution)
  - **Unbounded**
  - **Optimal solution**

- Phase I
  - We construct an **artificial linear programming problem** and solve it by the simplex method
  - Determine whether the original prob. is **feasible or infeasible**
  - If feasible, obtain a basic feasible solution of the original prob.

- Phase II
  - Using the feasible solution obtained in Phase I as an **initial solution**, solve the **original problem** by the simplex method (We obtain the optimal solution)

Two-Phase Method (Phase I) (Example)

- Original problem (standard form)

$$\begin{aligned} &\text{mimimize} \quad z = -x_1 - 5x_2 \\ &\text{subject to} \quad 4x_1 - x_2 + 4x_3 = 6 \\ &\qquad x_1 + 2x_2 + 2x_3 = 4 \\ &\qquad x_1, x_2, x_3 \geqq 0 \end{aligned}$$

Initial basic feasible solution ???

- Artificial problem

$$\begin{aligned} &\text{mimimize} \quad w = x_4 + x_5 \\ &\text{subject to} \quad 4x_1 - x_2 + 4x_3 + x_4 = 6 \\ &\qquad x_1 + 2x_2 + 2x_3 + x_5 = 4 \\ &\qquad x_1, x_2, x_3, x_4, x_5 \geqq 0 \end{aligned}$$

Introduce one **artificial variable** for each constraint

Minimize the sum of **artificial variables** (i.e., we want **all artificial variables** to become **0**)

Initial basic feasible solution

$(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 6, 4)$

Apply the simplex method

Optimal solution of the artificial problem $(0, 2/5, 8/5, 0, 0)$ This optimal solution is feasible for the original problem

Two-Phase Method (Phase I)

- minimize $z = \sum_{j=1}^{n} c_j x_j$

  subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i \quad (i = 1, 2, \ldots, m)$

  $x_j \geqq 0 \quad (j = 1, 2, \ldots, n)$

Artificial problem

- minimize $w = \sum_{i=1}^{m} x_{n+i}$

  subject to $\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \quad (i = 1, 2, \ldots, m)$

  $x_j \geqq 0 \quad (j = 1, 2, \ldots, n+m)$

Introduce one **artificial variable** for each constraint

Minimize the sum of **artificial variables**

The artificial problem **always** has a basic feasible solution $(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = (0, 0, \ldots, 0, b_1, \ldots, b_m)$

Two-Phase Method (Phase I)

- Optimal value w^* of the artificial problem
- **Case 1**: $w^* > 0$
  - Artificial variables cannot be reduced to 0
  - The original problem is **infeasible**
- **Case 2**: $w^* = 0$,
  no artificial variable is basic
  - All artificial variables are 0
  - The optimal solution of Phase I corresponds to a **basic feasible solution** of the original problem
    → An initial basic feasible solution of Phase II is obtained
- Case 3: $w^* = 0$,
  but some artificial variable remain basic
  - Consult the final dictionary of Phase I

Two-Phase Method (Phase II) (Example)

- Original problem (standard form)

$$\begin{aligned} &\text{mimimize } z = -x_1 - 5x_2 \\ &\text{subject to } 4x_1 - x_2 + 4x_3 = 6 \\ &\qquad x_1 + 2x_2 + 2x_3 = 4 \\ &\qquad x_1, x_2, x_3 \geqq 0 \end{aligned}$$

- Artificial problem

$$\begin{aligned} &\text{mimimize } w = x_4 + x_5 \\ &\text{subject to } 4x_1 - x_2 + 4x_3 + x_4 = 6 \\ &\qquad x_1 + 2x_2 + 2x_3 + x_5 = 4 \\ &\qquad x_1, x_2, x_3, x_4, x_5 \geqq 0 \end{aligned}$$

Initial basic feasible solution

		x_1
z	-2	-3
x_3	8/5	-9/10
x_2	2/5	2/5

- Remove all artificial variables
- Restore the original objective function z

Optimal solution of the artificial problem

		x_1	x_5	x_4
w	0	0	1	1
x_3	8/5	-9/10	-1/10	-1/5
x_2	2/5	2/5	-2/5	1/5

First, make sure you understand Cases 1 and 2

Phase I, Case 3 (Example)

mimimize $w = x_4 + x_5$

subject to

$$- x_2 - 2x_3 + x_4 = 0$$
$$x_1 + 3x_2 + 4x_3 + x_5 = 5$$
$$x_1, x_2, x_3, x_4, x_5 \geqq 0$$

Optimal solution of the artificial problem

		x_2	x_3	x_5
w	0	1	2	1
x_4	0	1	2	0
x_1	5	-3	-4	-1

Unfortunately, artificial variable x_4 is a basic variable

- Make a variable of the original problem a basic variable

  Ex.) pivot on x_2 to remove x_4 from the basis

  $x_4 = 0 + x_2 + 2x_3$

  $x_2 = 0 + x_4 - 2x_3$

		x_4	x_3	x_5
w	0	1	0	1
x_2	0	1	-2	0
x_1	5	-3	2	-1

First, make sure you understand Cases 1 and 2

Phase I, Case 3

- Dictionary
  - Artificial variable: $x_i = b_i + a_{i,j_1} x_{j_1} + a_{i,j_2} x_{j_2} + \ldots + a_{i,j_n} x_{j_n}$

Unfortunately, an artificial variable remain basic

- Case 3-1:

Make a variable of the original problem a basic variable

  - If there exists a variable x_{j_k} from the original problem such that $a_{i,j_k} \neq 0$, then pivot on x_{j_k} to remove x_i from the basis
    - $x_{j_k} = b_i / a_{i,j_k} + a_{i,j_1} / a_{i,j_k} x_{j_1} + \ldots$
- Case 3-2:

x_i depends only on artificial variables

  - If $a_{i,j_k} = 0$ holds for all original variables x_{j_k}, then x_i depend only on artificial variables
    → We can **remove** the constraint for x_i

- If Assumption 2 (rank(A) = m) does not hold, some constraint can be eliminated by Case 3-2
- If Assumption 1 ($n \geqq m$) does not hold (i.e, $n < m$), then we have rank(A) < m, and at least m – n constraints can be eliminated

Practice： Two-Phase Method

a. The following is the standard form of the problem in "Optimi-zation Technique (3)" slide p. 10. Solve the problem by the two-phase method. (First, introduce an artificial variable x_4)

$$\begin{aligned} &\text{mimimize } \; z = -x_1 - 2x_2 \\ &\text{subject to } \; -x_1 - x_2 - x_3 = 1 \\ &\qquad\qquad\quad x_1, x_2, x_3 \geqq 0 \end{aligned}$$

Practice：

- Transform the following problems into their standard forms, and solve them by the two-phase method
  - b. "Optimi-zation Technique (3)" slide p. 9
  - c. "Optimi-zation Technique (3)" slide p. 11

Practice： Two–Phase Method

(a) Solve the following problem by the two-phase method (First, introduce artificial variables)

$$
\begin{aligned}
\text{mimimize } \; z = & \; -x_1 - 2x_2 \\
\text{subject to } \; & \; -x_1 - x_2 = 2 \\
& \; -2x_1 - x_2 = 8 \\
& \; x_1, x_2 \geqq 0
\end{aligned}
$$

(b) Transform the following problem into its standard form, and solve it by the two-phase method

$$
\begin{aligned}
\text{maximize } \; z = & \; -3x_1 - x_2 \\
\text{subject to } \; & \; x_1 + x_2 \leqq 4 \\
& \; x_1 + 2x_2 = 5 \\
& \; x_1, x_2 \geqq 0
\end{aligned}
$$

Practice： Two-Phase Method

- Transform the following problems into their standard form, and solve them by the two-phase method

(a)

$$\begin{aligned} \text{maximize } z = &\; x_1 + 4x_2 \\ \text{subject to } &\; 2x_1 + x_2 \leqq 8 \\ &\; x_1 + 2x_2 \leqq 10 \\ &\; x_1, x_2 \geqq 0 \end{aligned}$$

(b)

$$\begin{aligned} \text{maximize } z = &\; x_1 + 4x_2 \\ \text{subject to } &\; 2x_1 + x_2 \geqq 8 \\ &\; x_1 + 2x_2 \geqq 10 \\ &\; x_1, x_2 \geqq 0 \end{aligned}$$

Summary

Problem in p. 10 (b)

$$\begin{aligned} \text{maximize} \quad & z = -3x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 \leqq 4 \\ & x_1 + 2x_2 = 5 \\ & x_1, x_2 \geqq 0 \end{aligned}$$

↓ Transform into its **standard form**

$$\begin{aligned} \text{minimize} \quad & z = 3x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 4 \\ & x_1 + 2x_2 = 5 \\ & x_1, x_2, x_3 \geqq 0 \end{aligned}$$

↓ Two-phase method Step 1: **artificial prob.**

$$\begin{aligned} \text{miniimize} \quad & w = x_4 + x_5 \\ \text{subject to} \quad & x_1 + x_2 + x_3 + x_4 = 4 \\ & x_1 + 2x_2 + x_5 = 5 \\ & x_1, x_2, x_3, x_4, x_5 \geqq 0 \end{aligned}$$

		x_1	x_2	x_3
w	9	-2	-3	-1
x_4	4	-1	-1	-1
x_5	5	-1	-2	0

↓ Solve the **artificial problem** by the simplex method

		x_1	x_5	x_4
w	0	0	1	1
x_3	3/2	-1/2	1/2	-1
x_2	5/2	-1/2	-1/2	0

Artificial variables are 0s

↓ Step 2: Initial basic feasible solution

		x_1
z	**5/2**	**5/2**
x_3	3/2	-1/2
x_2	5/2	-1/2