Large-Scale Knowledge Processing
Optimization Techniques (3)
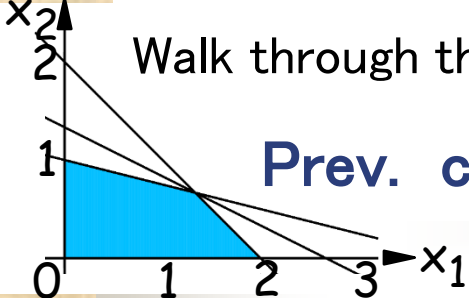
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Prev. class +α : Simplex method

$$\text{mimimize } z = -4x_1 - 5x_2$$
$$\text{subject to} \quad 2x_1 + 2x_2 + x_3 \qquad = 4$$
$$3x_1 + 6x_2 + \quad x_4 \quad = 8$$
$$x_1 + 4x_2 + \qquad x_5 = 4$$
$$x_1, x_2, x_3, x_4, x_5 \geqq 0$$

Initial feasible basic solution

		x_1	x_2
z	0	- 4	- 5
x_3	4	- 2	- 2
x_4	8	- 3	- 6
x_5	4	- 1	- 4

$$z = -4x_1 - 5x_2$$
$$x_3 = -2x_1 - 2x_2 + 4$$
$$x_4 = -3x_1 - 6x_2 + 8$$
$$x_5 = - \quad x_1 - 4x_2 + 4$$

Pivot
(3, 2)

		x_1	x_5
z	-5	-11/4	5/4
x_3	2	- 3/2	1/2
x_4	2	- 3/2	- 3/2
x_2	1	- 1/4	- 1/4

Increase x_2 from 0
(Fix other nonbasic var. $x_1 = 0$)

Pivot
(1, 1)

x_3 decrease, $x_3 \geqq 0$ if $x_2 \leqq 2$
x_4 decrease, $x_4 \geqq 0$ if $x_2 \leqq 4/3$
x_5 decrease, $x_5 \geqq 0$ if $x_2 \leqq 1$

		x_3	x_5
z	-26/3	11/6	1/3
x_1	4/3	- 2/3	1/3
x_4	0	1	1
x_2	2/3	1/6	- 1/3

$$x_2 = -1/4 x_1 - 1/4 x_5 + 1$$

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Initial **_feasible_** basic solution ?

- General way will be given later
- Easy case :

maximize $z = 4 x_1 + 6 x_2$
subject to $2 x_1 + 2 x_2 \leqq 4$
             $3 x_1 + 6 x_2 \leqq 9$
             $x_1, x_2 \geqq 0$

➡️

minimize $z = -4 x_1 - 6 x_2$
subject to $2 x_1 + 2 x_2 + x_3 = 4$
             $3 x_1 + 6 x_2 + x_4 = 9$
             $x_1, x_2, x_3, x_4 \geqq 0$

Transformation to its standard form

- Basic vars. : x_3, x_4
  - $z = -4 x_1 - 6 x_2$
  - $x_3 = 4 - 2 x_1 - 2 x_2$
  - $x_4 = 9 - 3 x_1 - 6 x_2$

Choice of pivot ?

- One of the typical ways :
  select variable x_j whose coefficient c_j is $c_j < 0$
  and $|c_j|$ is the largest

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Tips: Pivot operation

just update the simplex tableau

		x_1	x_2
z	0	- 4	- 6
x_3	4	- 2	- 2
x_4	9	- 3	- 6

		x_1	x_4
z			
x_3			
x_2	3/2	- 1/2	- 1/6

9 / -(- 6)   1 / (- 6)

Ex.: Pivot (2, 2)

Exchange the var. in the **2nd row** and the var. in the 2nd column

2nd row

$x_4 = b_4 + a_1 x_1 + \underline{a_2 x_2}$

$x_2 = b_4/(-a_2) + a_1/(-a_2) x_1 + 1/a_2 x_4$

Divide all elements in the **2nd row** by $-a_2$, but $1/a_2$ in the 2nd column

just update the simplex tableau

	x_1	x_2
z	0	- 4	- 6
x_3	4	- 2	- 2
x_4	9	- 3	- 6

	x_1	x_4
z			
x_3			
x_2	3/2	- 1/2	- 1/6

4 + (3/2)(**- 2**)

	x_1	x_4
z			
x_3	1	- 1	1/3
x_2	3/2	- 1/2	- 1/6

(- 1/6)(**- 2**)

Ex.: Pivot (**2** , 2)

Exchange the var. in the **2nd row** and the var. in the 2nd column

Other rows

$x_3 = b_3 + a_1 x_1 + \underline{a_2 x_2}$

$x_2 = b'_2 + a'_1 x_1 + a'_4 x_4$

(Eq. on x_2) × a_2 + (Eq. on x_3)

i.e., i-th column becomes $a_i a_2 + a'_i$

Notice: 2nd column: (Eq. x_2) × a_2
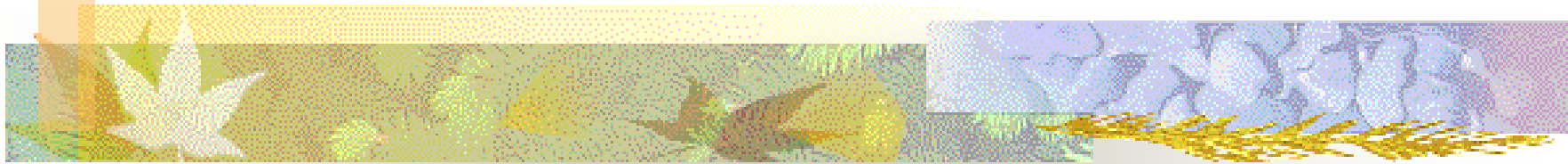
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Solve the following linear programming problems by simplex method

a. "Optimization Techniques (2)" slide p. 24
b. "Optimization Techniques (2)" slide p. 13
c. "Optimization Techniques (2)" slide p. 15
d. "Optimization Techniques (1)" slide p. 7

Linear programming
Simplex method

- Can we solve all linear programming problem ?

Original problem

$$\text{mimimize } z = -x_1 - 2x_2$$
$$\text{subject to} \quad x_1 + x_2 \geqq -1$$
$$x_1, x_2 \geqq 0$$

Initial feasible basic solution

		x_1	x_2
z	0	-1	-2
x_3	1	1	1

Standard form

$$-x_1 - x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geqq 0$$

Increase x_2 from 0
(Fix other nonbasic variables as $x_1 = 0$)

x_3 increases, always satisfies $x_3 \geqq 0$

→ Objective value z decreases

Feasible region is **unbounded**
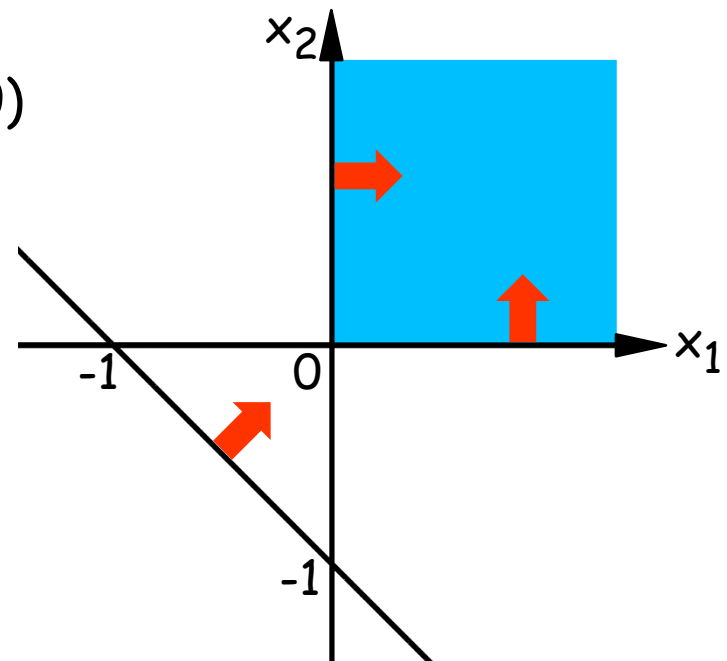
Situation 2： Infeasible （no feasible solutions）

Original problem

$$\text{mimimize } z = -x_1 - 2x_2$$
$$\text{subject to} \quad x_1 + x_2 \leqq -1$$
$$x_1, x_2 \geqq 0$$

Initial feasible basic solution

	x_1	x_2
z	0	-1	-2
x_3	-1	-1	-1

Not a feasible basic solution

Standard form

$$-x_1 - x_2 - x_3 = 1$$
$$x_1, x_2, x_3 \geqq 0$$

How to decide the problem is infeasible ???

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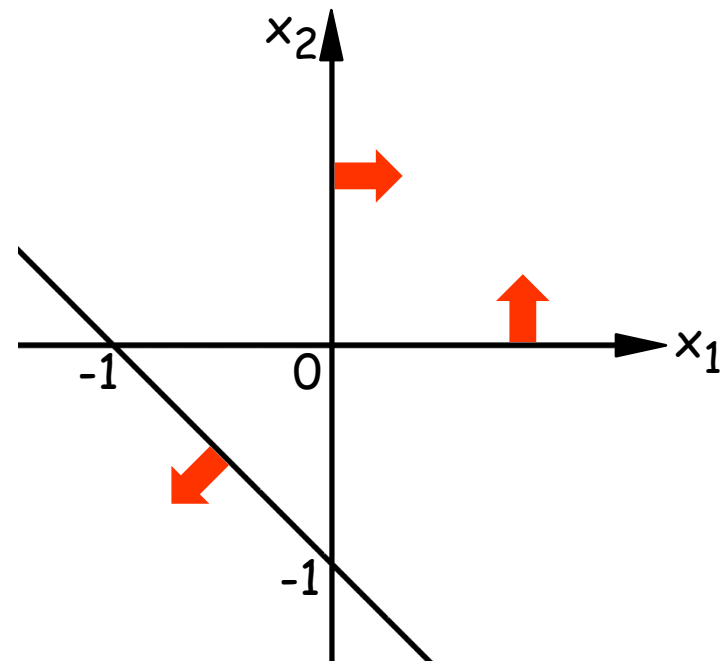
Situation 3: Initial feasible basic solution

Original problem

$$\text{mimimize } z = -x_1 - 2x_2$$
$$\text{subject to} \quad x_1 + x_2 \geqq 1$$
$$x_1, x_2 \geqq 0$$

Initial feasible basic solution

		x_1	x_2
z	0	-1	-2
x_3	-1	1	1

Standard form

$$x_1 + x_2 - x_3 = 1$$
$$x_1, x_2, x_3 \geqq 0$$

Not a feasible basic solution
(The right fig. says $(0, 0)$ is not a feasible solution, but we have feasible region)

		x_1	x_3
z	0	1	-2
x_2	1	-1	1

$(0, 1)$ is a feasible basic solution

How to find an initial feasible basic solution ???
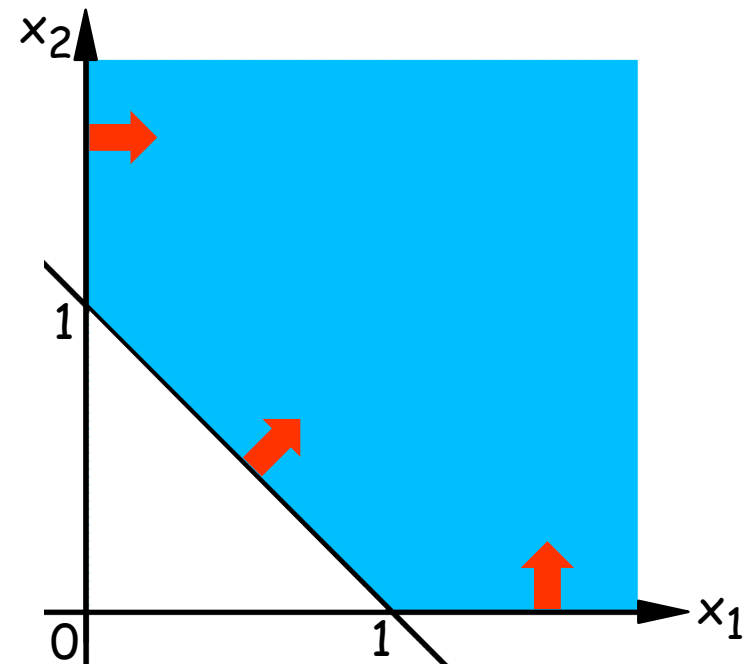
Two-phase simplex method

■ Original problem (in standard form)

minimize $z = -x_1 - 5x_2$
subject to $\quad 4x_1 - x_2 + 4x_3 = 6$
$\qquad\qquad x_1 + 2x_2 + 2x_3 = 4$
$\qquad\qquad x_1, x_2, x_3 \geqq 0$

How to find an initial feasible basic solution ???

Introduce an **artificial variable** for each eq.

Minimize the sum of the **artificial vars.**
(so that **all artificial vars.** are **0s.**)

■ Artificial problem

minimize $w = \mathbf{x_4} + \mathbf{x_5}$
subject to $\quad 4x_1 - x_2 + 4x_3 + \mathbf{x_4} \qquad = 6$
$\qquad\qquad x_1 + 2x_2 + 2x_3 \qquad + \mathbf{x_5} = 4$
$\qquad\qquad x_1, x_2, x_3, \mathbf{x_4}, \mathbf{x_5} \geqq 0$

Phase I: apply simplex method to the artificial problem

Initial feasible basic solution
$(x_1, x_2, x_3, \mathbf{x_4}, \mathbf{x_5})$
$= (0, 0, 0, \mathbf{6}, \mathbf{4})$

Optimal solution of the artificial problem
$(0, 2/5, 8/5, \mathbf{0}, \mathbf{0})$

This solution satisfies the constraints of the original prob.
= This is the initial feasible basic solution for the original prob.

Phase II: apply simplex method to the original prob.

Formulation of linear programming problems
（additional techniques）

max{ } function

- Minimize the maximum of two or more objective functions

  - minimize max{ f(x), g(x), h(x) }

  $\rightarrow$ Introduce a new variable z

  minimize   z
  subject to $f(x) \leqq z, \ g(x) \leqq z, \ h(x) \leqq z$

Practice：

- minimize $x_1 + |x_2|$

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max{ } function

- Minimize the maximum of two or more objective functions

  - minimize $\max\{ f(x), g(x), h(x) \}$

  $\rightarrow$ Introduce a new variable z

  minimize z
  subject to $f(x) \leqq z, \ g(x) \leqq z, \ h(x) \leqq z$

Practice :

$|x_2| = \max\{ x_2, -x_2 \}$

- minimize $x_1 + |x_2|$

$\rightarrow$ minimize $x_1 + x_2'$
subject to $x_2 \leqq x_2', \ -x_2 \leqq x_2'$

■ We have 2 variables x and y, where x = 0, 1, and y is a real number with y $\geqq$ 0. Show constraint inequalities for each of the following constraints

1. y $\geqq$ 30 when x = 1

2. y = 0 when x = 0

3. y $\leqq$ 10 when x = 0

4. y $\leqq$ 10 when x = 0, and y $\geqq$ 30 when x = 1

5. y $\leqq$ 10 when x = 0, and y $\leqq$ 30 when x = 1

Constraints (Mixed integer programming prob.)

1. $y \geqq 30$ when $x = 1$
   - **$y \geqq 30\,x$**
     i.e., **$-\,30\,x + y \geqq 0$**
   - When $x = 0$, this inequality says $y \geqq 0$ (thus, we have no constraint in this case)
   - When $x = 1$, we add constraint $y \geqq 30$

Constraints (Mixed integer programming prob.)

2. y = 0 when x = 0

   - **$y \leqq M\,x$**   (M is a sufficiently large constant)
         i.e., **$-M\,x + y \leqq 0$**

   - When x = 0, we have $y \leqq 0$.
     As we have $y \geqq 0$, we have constraint y = 0

3. $y \leqq 10$ when x = 0

   - **$-M\,x + y \leqq 10$**   (M is a sufficiently large constant)

Constraints (Mixed integer programming prob.)

4. $y \leqq 10$ when $x = 0$, $y \geqq 30$ when $x = 1$

  - **$-Mx + y \leqq 10$,** (M is a sufficiently large constant)
    $-30x + y \geqq 0$

  - From the above constraints 1 and 3

5. $y \leqq 10$ when $x = 0$, $y \leqq 30$ when $x = 1$

  - **$y \leqq 10(1 - x) + 30x$**
    i.e., **$-20x + y \leqq 10$**

We are given n points (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) in the plane. Each point is colored in red or blue.

1. We'd like to obtain a parabola that is convex at the bottom, such that **all red points** are **above** the parabola and **all blue points** are **below** the parabola. Formulate the problem of **finding the equation of such a parabola** (if it exists), as a linear programming problem.

2. We'd like to obtain a circle such that all **red points** are **inside the circle** (including the boundary) and all **blue points** are **outside the circle** (including the boundary) Formulate the problem of **determining** whether such a **circle exists** or not, as a linear programming problem.

Formulation (1)

- Without loss of generality, we have

  k red points (x_1, y_1), ..., (x_k, y_k), and

  n – k blue points (x_{k+1}, y_{k+1}), ..., (x_n, y_n)

- Let $y = a\,x^2 + b\,x + c$ denote the parabola

- Formulation is as follows:

  - minimize 0
  - subject to

Determine whether we have a feasible solution (a, b, c) satisfying the constraints

Red points： $x_i^2\, a + x_i\, b + c \geqq y_i$ (i = 1, 2, ..., k)

Blue points： $x_i^2\, a + x_i\, b + c \leqq y_i$ (i = k+1, k+2, ..., n)

$a > 0,\ b, c$：free variables

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Formulation (2)

- Without loss of generality, we have
  k red points (x_1, y_1), ..., (x_k, y_k), and
  n – k blue points (x_{k+1}, y_{k+1}), ..., (x_n, y_n)

- Let (x, y) and r denote the center
 and the radius of the circle, respectively
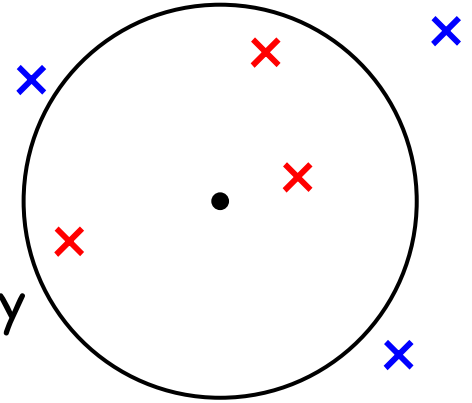
- Constraints for each point :

  Red points : $(x_i - x)^2 + (y_i - y)^2 \leqq r^2$ (i = 1, 2, ..., k)

  Blue points : $(x_i - x)^2 + (y_i - y)^2 \geqq r^2$ (i = k+1, k+2, ..., n)

- As we have x^2, y^2, r^2 in the constraints, we cannot formulate the problem as a linear programming problem ...   (is that right ?)

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Formulation (2)

- Let point C (x, y) be the center of the circle

- We try to denote the constraints in another way

- Any pair of **red point P_i (x_i, y_i)** and **blue point P_j (x_j, y_j)** satisfies **distance C $P_i \leqq$ distance C P_j**

- That is, for any $i = 1, 2, ..., k$ and $j = k+1, k+2, ..., n$, we have a constraint
$$(x_i - x)^2 + (y_i - y)^2 \leqq (x_j - x)^2 + (y_j - y)^2$$

- From this inequality, we have a constraint
$$2(x_i - x_j)x + 2(y_i - y_j)y \geqq x_i^2 + y_i^2 - x_j^2 - y_j^2$$
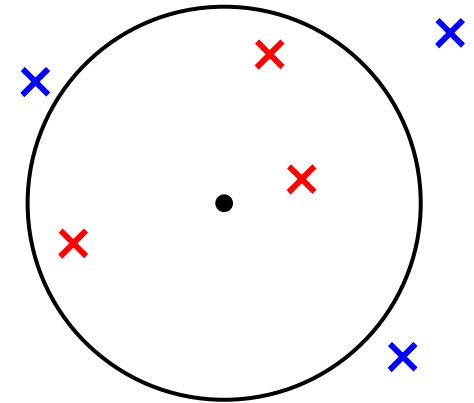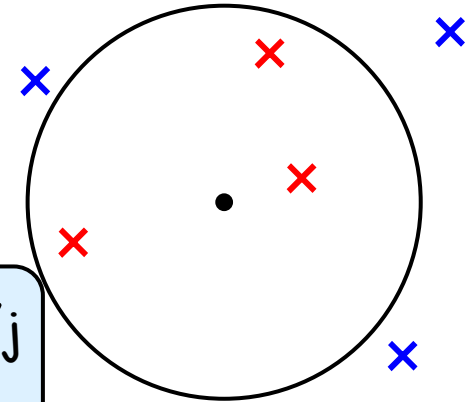
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Formulation (2)

- Formulated linear programming problem is as follows:

Determine whether we have a feasible solution (a, b, c) satisfying the constraints

- minimize   0

Note that x_i, y_i, x_j, y_j are constants

- subject to
$$2 (x_i - x_j) x + 2 (y_i - y_j) y \geqq x_i^2 + y_i^2 - x_j^2 - y_j^2$$
$$(i = 1, 2, ..., k; \ j = k+1, k+2, ..., n)$$

※ By solving this linear programming problem, we can obtain the center (x, y) of the circle. The radius is not obtained.

Summary

- Simplex method
  - How to find an initial feasible basic solution ?
  - How to select a pivot ?
- Situations we need to consider
  → Two-phase simplex method
- Formulation of linear programming problems