Large-Scale Knowledge Processing

Optimization Techniques (4)

Supplementary Material:

Complementary Slackness Theorem

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Complementary Slackness Theorem

- Let x and y be feasible solutions of the primal problem (P) and the dual problem (D), respectively
- x and y are optimal solutions
  $\Leftrightarrow$ the following conditions hold simultaneously
  - $y_i = 0$ or $a_i^T x = b_i$ $(i = 1, 2, \ldots, m)$
  - $y^T A_j = c_j$ or $x_j = 0$ $(j = 1, 2, \ldots, n)$

Proof) This follows directly from the Weak Duality Theorem

$$\sum_i b_i y_i = \sum_i \left(\sum_j a_{ij} x_j\right) y_i = \sum_j \left(\sum_i a_{ij} y_i\right) x_j \leqq \sum_j c_j x_j$$

Since x and y are optimal solutions, the equalities must hold.

Primal Problem (P)	
minimize	$z = c^T x$
subject to	$A x = b$
	$x \geqq 0$

Dual Problem (D)	
maximize	$w = y^T b$
subject to	$y^T A \leqq c^T$
	y^T free variables

Complementary Slackness Theorem

- Primal Problem

$$\begin{array}{ll} \text{mimimize} & z = -2x_1 + x_2 + x_3 - x_4 \\ \text{subject to} & x_1 + 2x_2 - x_3 + x_4 = 0 \\ & 2x_1 - 2x_2 + 3x_3 + 4x_4 = 9 \\ & x_1, x_2, x_3, x_4 \geqq 0 \end{array}$$

  - Optimal solution: (9/5, 0, 9/5, 0)
  - Optimal value: -9/5

$x_1, x_3 \neq 0$

- Dual Problem

$$\begin{array}{ll} \text{maximize} & w = 9y_2 \\ \text{subject to} & y_1 + 2y_2 \leqq -2 \\ & 2y_1 - 2y_2 \leqq 1 \\ & -y_1 + 3y_2 \leqq 1 \\ & y_1 + 4y_2 \leqq -1 \\ & y_1, y_2 : \text{ free variables} \end{array}$$

The 1st and 3rd constraints

  - $y_1 + 2y_2 = -2$
  - $-y_1 + 3y_2 = 1$ (- 8/5, - 1/5)

Optimal value: -9/5

Practice： Complementary Slackness Thm.

- The following are primal problems together with their optimal solutions and optimal values. For each case, formulate the corresponding dual problem, and using the complementary slackness theorem, determine its optimal solution and optimal value.

(a)

$$\begin{aligned} \text{mimimize } \; & z = -5x_1 + x_2 + x_3 - x_4 \\ \text{subject to } \; & 5x_1 + 2x_2 - x_3 + 3x_4 = 0 \\ & 2x_1 - 2x_2 + 2x_3 + 2x_4 = 9 \\ & x_1, x_2, x_3, x_4 \geqq 0 \end{aligned}$$

- Optimal solution： (3/4, 0, 15/4, 0)   Optimal value： 0

(b)

$$\begin{aligned} \text{mimimize } \; & z = 2x_1 - 3x_2 - 2x_3 + 5x_4 \\ \text{subject to } \; & 3x_1 - x_2 + 2x_3 + 3x_4 = 8 \\ & x_1 + 3x_2 - 2x_3 + 2x_4 = 10 \\ & x_1, x_2, x_3, x_4 \geqq 0 \end{aligned}$$

- Optimal solution： (0, 9, 17/2, 0)   Optimal value： -44