Large-scale Knowledge Processing
Lecture 10

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Today's Lecture

Study Boolean functions and its representations.

- Boolean Function
- Truth Table
- Boolean Formula (CNF, DNF)
- Karnaugh Map
- Boolean Circuit
- Binary Decision Tree → Next Lecture
- BDD (Binary Decision Diagram) → Next Lecture

Boolean Fucntion and Truth Table

Boolean Function

A Boolean Variable takes 0（false）or 1（true）value.

Boolean Operations:

- OR：$x \vee y$
- AND： $x \wedge y$
- NOT： $\bar{x}$
- Exclusive OR（EXOR，XOR）： $x \oplus y$

n-variate Boolean function $f: \{0,1\}^n \rightarrow \{0, 1\}$

Truth Table

A table presents all possible assignments to Boolean function f and their corresponding output values.

x	y	z	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

This example is a Majority function : Maj(x, y, z)

Maj(x, y, z) outputs 1 if $x+y+z \geq 2$

Truth tables of fundamental Boolean operations

AND function

x	y	$x \land y$
0	0	0
0	1	0
1	0	0
1	1	1

OR function

x	y	$x \lor y$
0	0	0
0	1	1
1	0	1
1	1	1

XOR function

x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

NOT function

x	$\neg x \ (= \bar{x})$
0	1
1	0

Boolean Formula

This is a representation of Boolean function by combining Boolean operations with literals.

For Boolean variable x, x is a positive literal and $\bar{x}$ is a negative literal.

（例1） ∧ is often omitted

$$f(x, y, z) = (x \vee y) \wedge (y \vee \bar{z}) \wedge (x \vee \bar{y} \vee z) \wedge \bar{x}$$

（例2）

$$g(x, y, z) = \big((x \oplus y) \wedge z\big) \oplus \big((x \vee \bar{y}) \oplus z\big)$$

Exercise 1

Complete the truth table of the following function.

$$g(x, y, z) = \big((x \oplus y) \land z\big) \oplus \big((x \lor \bar{y}) \oplus z\big)$$

x	y	z	f
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Exercise 1（How to solve）

Computing outputs of f by all input assignment

ex：$x = 0, y = 1, z = 1$

$$g(0,1,1) = \big((0 \oplus 1) \wedge 1\big) \oplus \big((0 \vee \bar{1}) \oplus 1\big)$$

$$= (1 \wedge 1) \oplus (0 \oplus 1) = 1 \oplus 1 = 0$$

Exercise 1 (Answer)

Complete the truth table of the following function.

$$g = \big((x \oplus y) \wedge z\big) \oplus \big((x \vee \overline{y}) \oplus z\big)$$

x	y	z	f
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

Boolean Formula

Converting a truth table to Boolean formula

How to convert a given truth table to a Boolean formula ?

x	y	z	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$\rightarrow \quad f = ???$

Converting a truth table to Boolean formula

How to convert a given truth table to a Boolean formula ?

⇒ Use disjunctive normal forms or conjunctive normal forms

Disjunctive Normal Form （DNF)

➢ OR of terms（term: AND of literals)

$$f(x, y, z) = \bar{x}y \vee x\bar{y}z \vee y$$

Conjunctive Normal Form （CNF)

➢ AND of clauses（clause：OR of literals)

$$f(x, y, z) = (x \vee z)y\,(\bar{y} \vee z)(\bar{x} \vee y \vee z)$$

The definition of Minterms

Minterms

- Minterm is the AND of various different literals in which each literal occurs exactly once. The output result of the minterm functions is 1

For example, we consider 3-variate function $f(x, y, z)$.

When $f(0,1,0) = 1$, $\bar{x}y\bar{z}$ is the miniterm of $f(0,1,0)$.

How to convert a given truth table to a DNF representation.

By the following expansion to minterms

$$f(x,y,z)$$

$$= \bar{x} \wedge f(0,y,z) \vee x \wedge f(1,y,z)$$

$$= \bar{x}\big(\bar{y} \wedge f(0,0,z) \vee y \wedge f(0,1,z)\big) \vee x\big(\bar{y} \wedge f(1,0,z) \vee y \wedge f(1,1,z)\big)$$

$$= \bar{x}\bar{y}\big(\bar{z} \wedge f(0,0,0) \vee z \wedge f(0,0,1)\big) \vee \bar{x}y\big(\bar{z} \wedge f(0,1,0) \vee z \wedge f(0,1,1)\big)$$

$$\vee\ xy\big(\bar{z} \wedge f(1,0,0) \vee z \wedge f(1,0,1)\big) \vee xy\big(\bar{z} \wedge f(1,1,0) \vee z \wedge f(1,1,1)\big)$$

$$= \bar{x}\bar{y}\bar{z}f(0,0,0) \vee \bar{x}\bar{y}zf(0,0,1) \vee \bar{x}y\bar{z}f(0,1,0) \vee \bar{x}yzf(0,1,1)$$

$$\vee\ x\bar{y}\bar{z}f(1,0,0) \vee x\bar{y}zf(1,0,1) \vee xy\bar{z}f(1,1,0) \vee xyzf(1,1,1)$$

How to convert a given truth table to a DNF representation.

We can easy to obtain the DNF representation of f by combining minterms with OR where the output of f is 1.

	x	y	z	f
$\bar{x}\bar{y}\bar{z}$	0	0	0	0
$\bar{x}\bar{y}z$	0	0	1	0
$\bar{x}y\bar{z}$	0	1	0	0
$\bar{x}yz$	0	1	1	1
$x\bar{y}\bar{z}$	1	0	0	0
$x\bar{y}z$	1	0	1	1
$xy\bar{z}$	1	1	0	1
xyz	1	1	1	1

$$f(x,y,z) = \bar{x}yz \vee x\bar{y}z \vee xy\bar{z} \vee xyz$$

The definition of Maxterms

Maxterms

- Maxterm is OR of various different literals in which each literal occurs exactly once. The output result of the minterm functions is 0

For example, we consider 3-variate function $f(x, y, z)$.

When $f(0,1,0) = 0$, $x \vee \bar{y} \vee z$ is the miniterm of $f(0,1,0)$.

How to convert a given truth table to a CNF representation.

By the following expansion to maxterms

$$f(x,y,z)$$

$$= (x \lor f(0,y,z))(\bar{x} \lor f(1,y,z))$$

$$= \Big(x \lor \big(y \lor f(0,0,z)\big)\big(\bar{y} \lor f(0,1,z)\big)\Big)\ \Big(\bar{x} \lor \big(y \lor f(1\ 0,z)\big)\big(\bar{y} \lor f(1,1,z)\big)\Big)$$

$$= \big(x\ \lor y \lor f(0,0,z)\big)\big(x \lor \bar{y} \lor f(0,1,z)\big)\big(\bar{x} \lor y \lor f(1\ 0,z)\big)\big(\bar{x} \lor \bar{y} \lor f(1,1,z)\big)$$

$$= \big(x\ \lor y \lor z \lor f(0,0,0)\big)\big(x\ \lor y \lor \bar{z} \lor f(0,0,1)\big)\big(x \lor \bar{y} \lor z \lor f(0,1,0)\big)$$

$$\land \big(x \lor \bar{y} \lor \bar{z} \lor f(0,1,1)\big)\big(\bar{x} \lor y \lor z \lor f(1\ 0,0)\big)\big(\bar{x} \lor y \lor \bar{z} \lor f(1\ 0,1)\big)$$

$$\land \big(\bar{x} \lor \bar{y} \lor z \lor f(1,1,0)\big)\big(\bar{x} \lor \bar{y} \lor \bar{z} \lor f(1,1,1)\big)$$

How to convert a given truth table to a CNF representation.

We can easy to obtain the CNF representation of f by combining maxterms with AND where the output of f is 0.

	x	y	z	f
$x \vee y \vee z$	0	0	0	0
$x \vee y \vee \bar{z}$	0	0	1	0
$x \vee \bar{y} \vee z$	0	1	0	0
$x \vee \bar{y} \vee \bar{z}$	0	1	1	1
$\bar{x} \vee y \vee z$	1	0	0	0
$\bar{x} \vee y \vee \bar{z}$	1	0	1	1
$\bar{x} \vee \bar{y} \vee z$	1	1	0	1
$\bar{x} \vee \bar{y} \vee \bar{z}$	1	1	1	1

$$f(x, y, z) = (x \vee y \vee z)(x \vee y \vee \bar{z})(x \vee \bar{y} \vee z)(\bar{x} \vee y \vee z)$$

Exercise 2

Converting the following truth table by a CNF and a DNF

x	y	z	f
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

Exercise 2（Answer）

Disjunctive Normal Form (DNF)

	x	y	z	f
$\bar{x}\bar{y}\bar{z}$	0	0	0	1
$\bar{x}\bar{y}z$	0	0	1	0
$\bar{x}y\bar{z}$	0	1	0	0
$\bar{x}yz$	0	1	1	0
$x\bar{y}\bar{z}$	1	0	0	1
$x\bar{y}z$	1	0	1	1
$xy\bar{z}$	1	1	0	1
xyz	1	1	1	0

$$f(x, y, z) = \bar{x}\bar{y}\bar{z} \vee x\bar{y}\bar{z} \vee x\bar{y}z \vee xy\bar{z}$$

Exercise 2（Answer）

Conjunctive Normal Form (CNF)

	x	y	z	f
$x \lor y \lor z$	0	0	0	1
$x \lor y \lor \bar{z}$	0	0	1	0
$x \lor \bar{y} \lor z$	0	1	0	0
$x \lor \bar{y} \lor \bar{z}$	0	1	1	0
$\bar{x} \lor y \lor z$	1	0	0	1
$\bar{x} \lor y \lor \bar{z}$	1	0	1	1
$\bar{x} \lor \bar{y} \lor z$	1	1	0	1
$\bar{x} \lor \bar{y} \lor \bar{z}$	1	1	1	0

$$f(x, y, z) = (x \lor y \lor \bar{z})(x \lor \bar{y} \lor z)(x \lor \bar{y} \lor \bar{z})(\bar{x} \lor \bar{y} \lor \bar{z})$$

Karnaugh Map

Karnaugh Map

The 2D table of all outputs of Boolean function.

- Adjacent input assignments differ by 1 bit.
- Each cell corresponds to maxterm of f.

$xy \backslash zw$	00	01	11	10
00	$\bar{x}\bar{y}\bar{z}\bar{w}$	$\bar{x}\bar{y}\bar{z}w$	$\bar{x}\bar{y}zw$	$\bar{x}\bar{y}z\bar{w}$
01	$\bar{x}y\bar{z}\bar{w}$	$\bar{x}y\bar{z}w$	$\bar{x}yzw$	$\bar{x}yz\bar{w}$
11	$xy\bar{z}\bar{w}$	$xy\bar{z}w$	$xyzw$	$xyz\bar{w}$
10	$\bar{x}y\bar{z}\bar{w}$	$\bar{x}y\bar{z}w$	$\bar{x}yzw$	$\bar{x}yz\bar{w}$

Karnaugh Map

The 2D table of all outputs of Boolean function.

- Adjacent input assignments differ by 1 bit.
- Use a Karnaugh map to simplify a Boolean function.

xy \ zw	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	0	1	1	0
10	0	1	1	0

xy \ zw	00	01	11	10
00	$\bar{x}\bar{y}\bar{z}\bar{w}$	$\bar{x}\bar{y}\bar{z}w$	$\bar{x}\bar{y}zw$	$\bar{x}\bar{y}z\bar{w}$
01	$\bar{x}y\bar{z}\bar{w}$	$\bar{x}y\bar{z}w$	$\bar{x}yzw$	$\bar{x}yz\bar{w}$
11	$xy\bar{z}\bar{w}$	$xy\bar{z}w$	$xyzw$	$xyz\bar{w}$
10	$\bar{x}y\bar{z}\bar{w}$	$\bar{x}y\bar{z}w$	$\bar{x}yzw$	$\bar{x}yz\bar{w}$

Simplify a Boolean function using Karnaugh map

We can simplify a Boolean function by using the fact that adjacent input assignments differ by 1 bit.

$$\bar{x}\bar{y}zw \lor \bar{x}\bar{y}z\bar{w} = \bar{x}\bar{y}z(w \lor \bar{w}) = \bar{x}\bar{y}z$$

xy \ zw	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	0	1	1	0
10	0	1	1	0

xy \ zw	00	01	11	10
00	$\bar{x}\bar{y}\bar{z}\bar{w}$	$\bar{x}\bar{y}\bar{z}w$	$\bar{x}\bar{y}zw$	$\bar{x}\bar{y}z\bar{w}$
01	$\bar{x}y\bar{z}\bar{w}$	$\bar{x}y\bar{z}w$	$\bar{x}yzw$	$\bar{x}yz\bar{w}$
11	$xy\bar{z}\bar{w}$	$xy\bar{z}w$	$xyzw$	$xyz\bar{w}$
10	$\bar{x}y\bar{z}\bar{w}$	$\bar{x}y\bar{z}w$	$\bar{x}yzw$	$\bar{x}yz\bar{w}$

Simplify a Boolean function using Karnaugh map

xy \ zw	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	0	1	1	0
10	0	1	1	0

These two input differs only with respect to the bit of w.

$\Rightarrow \bar{x}\bar{y}z$ is same

$\Rightarrow$ we can reduce two terms to one term.

Similary, we can reduce 2^k (k=1,2,⋯) adjacent terms to one term.

Simplify a Boolean function using Karnaugh map

The brown rectangle represents the term zw.

The blue rectangle represents the term xw.

xy \ zw	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	0	1	1	0
10	0	1	1	0

zw

xy \ zw	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	0	1	1	0
10	0	1	1	0

xw

How to simplification by Karnaugh map

For example, we consider the following Boolean function

$$f = \bar{x}\bar{y}zw \lor \bar{x}\bar{y}z\bar{w} \lor xy\bar{z}w \lor xyzw \lor x\bar{y}\bar{z}w \lor x\bar{y}zw$$

At first, we write the value of f in Karnaugh map.

xy \ zw	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	0	1	1	0
10	0	1	1	0

How to simplification by Karnaugh map

Next, we cover all 1s by some rectangles as the following.

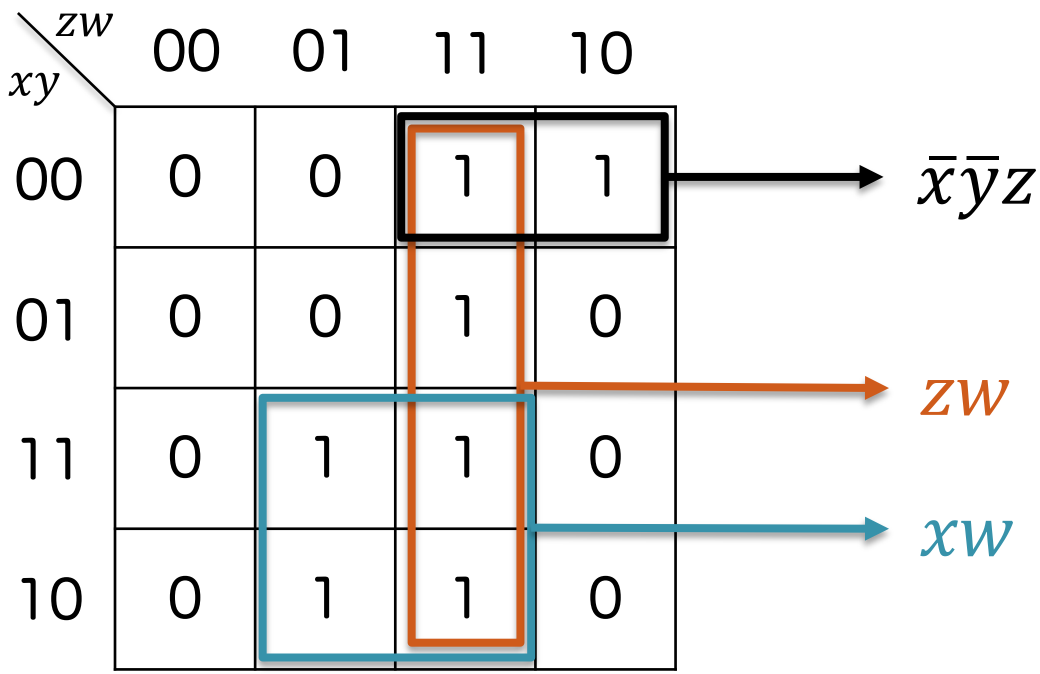
- Cannot cover any 0.
- The size of rectangle must be $2^i \times 2^j$ $(0 \leq i, j)$

xy \ zw	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	0	1	1	0
10	0	1	1	0

How to simplification by Karnaugh map

We calculate the term corresponding to each rectangle.

Combining these terms by OR, we can simplify f

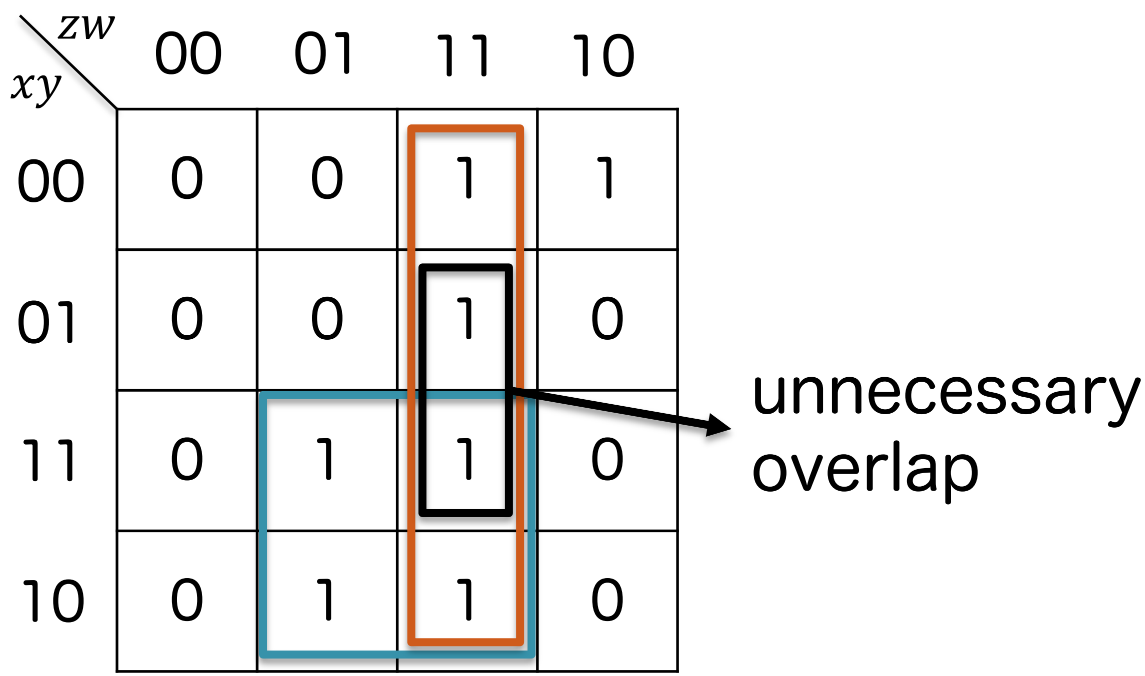


$$f = \bar{x}\bar{y}z \vee xw \vee zw$$

Note for simplifications

The following is important for simplifications.

- ➢ Take a large adjacent cells as possible
- ➢ No unnecessary overlap



Exercise 3

Simplify the following Boolean function by Karnaugh map

$$f(x,y,z,w) = xyzw \vee xy\bar{z}w \vee xyz\bar{w} \vee xy\bar{z}\bar{w}$$
$$\vee\ \bar{x}yz\bar{w} \vee \bar{x}y\bar{z}\bar{w} \vee\ \bar{x}\bar{y}z\bar{w} \vee \bar{x}\bar{y}\bar{z}\bar{w}$$

xy \ zw	00	01	11	10
00				
01				
11				
10				

Exercise 3（Answer)

First, complete the Karnaugh map of f

$$f(x,y,z,w) = xyzw \vee xy\bar{z}w \vee xyz\bar{w} \vee xy\bar{z}\bar{w} \vee \bar{x}yz\bar{w} \vee \bar{x}y\bar{z}\bar{w} \vee \bar{x}\bar{y}z\bar{w} \vee \bar{x}\bar{y}\bar{z}\bar{w}$$

xy \ zw	00	01	11	10
00	1	0	0	1
01	1	0	0	1
11	1	1	1	1
10	0	0	0	0

Exercise 3（Answer)

Simplify f by the Karnaugh map.

xy \ zw	00	01	11	10
00	1	0	0	1
01	1	0	0	1
11	1	1	1	1
10	0	0	0	0

$\bar{x}\bar{w}$

xy

$$f(x, y, z, w) = \bar{x}\bar{w} \vee xy$$

Boolean Circuit

Boolean Circuit

A Circuit deal with Boolean values.

Constructed by some Boolean gates.

Combinatorial Circuit

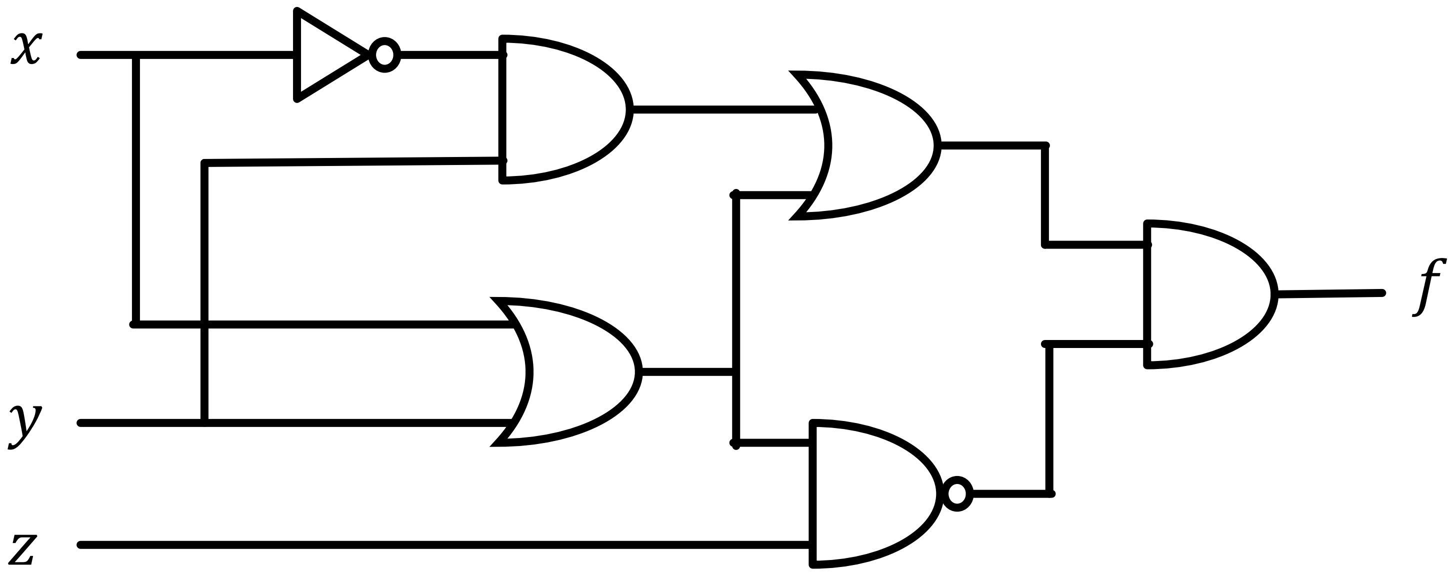
- Output doesn’t depend on the previous inputs.
- Any Boolean function can be written by a circuit.

Sequential Circuit

- Using the previous output of the circuit by Flip-flops (some kind of memories).

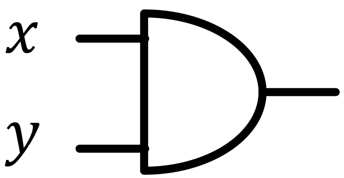
An example of a Boolean Circuit

$$f = \left((\bar{x} \wedge y) \vee (x \vee y) \right) \wedge \left(\overline{(x \vee y) \wedge z} \right)$$

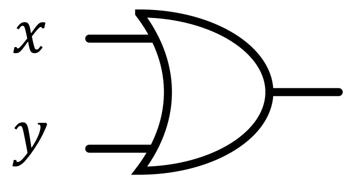


Boolean gates

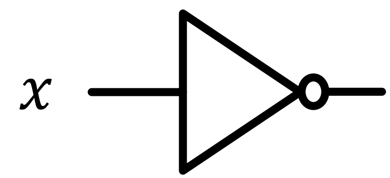
Examples



AND gate



OR gate



NOT gate

x	y	$x \land y$
0	0	0
0	1	0
1	0	0
1	1	1

x	y	$x \lor y$
0	0	0
0	1	1
1	0	1
1	1	1

x	$\neg x\ (= \bar{x})$
0	1
1	0

Completeness

Complete Set

- If we can construct a circuit for any Boolean function by a set of Boolean gates S, S is a complete set.
- A set of Boolean gates S is complete set, S satisfies completeness.

NAND

x	y	$\overline{x \land y}$
0	0	1
0	1	1
1	0	1
1	1	0

Examples of a Complete Set

- S={AND、OR、NOT}
- S={NAND}

Completeness

The reason why {AND、OR、NOT} is a complete set.

- Any Boolean function can be written by DNF or CNF.
- DNF and CNF use only AND, OR, and NOT gates.

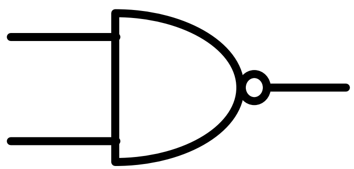
The reason why {NAND} is a complete set.

- AND, OR, and NOT can be represented by only NAND.

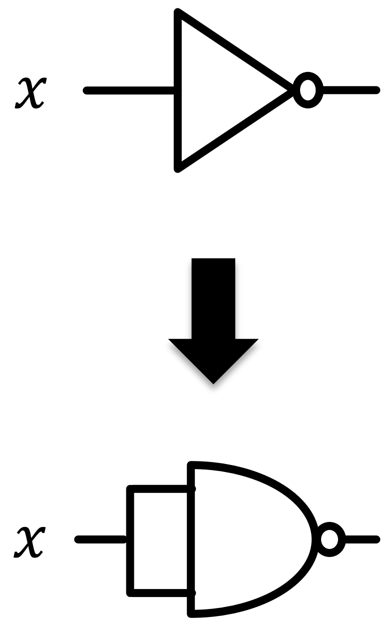
Exercise 4

NOT gate can be represented by only NAND gate as follows. Represent AND gate and OR gate by only NAND gate.

NAND gate



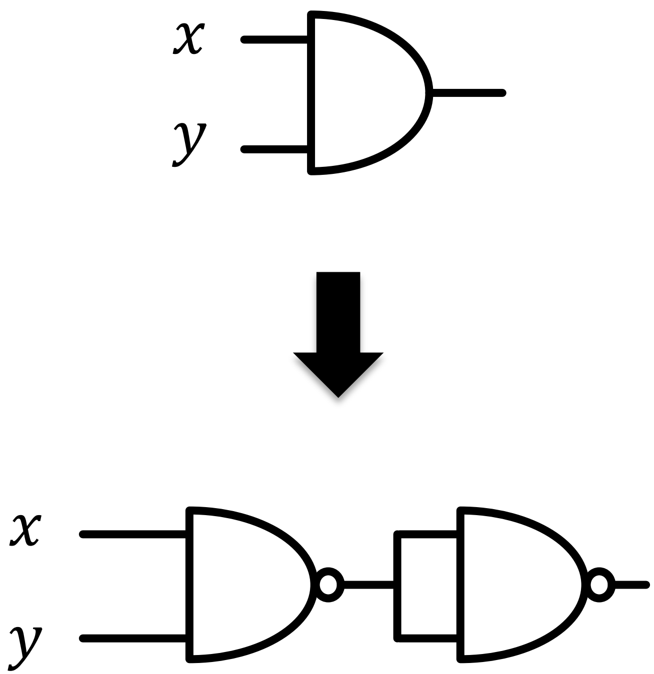
x	y	$\overline{x \wedge y}$
0	0	1
0	1	1
1	0	1
1	1	0

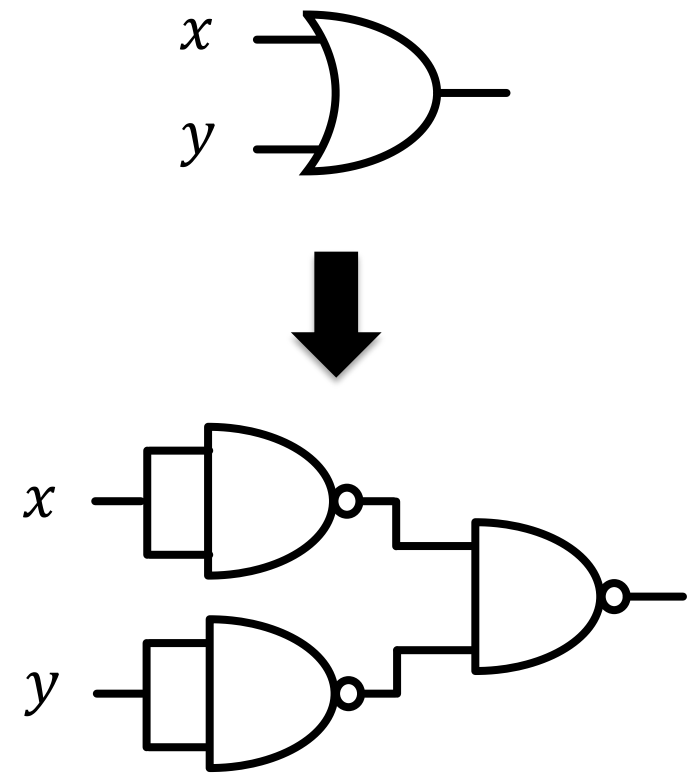


Exercise 4（Answer）

AND: NOT of AND is NAND.

OR: By DeMoragan law, $x \vee y = \overline{\overline{x} \wedge \overline{y}}$.





Summary

Introduce Boolean functions and their representations.

- Boolean Function
- Truth Table
- Boolean Formula（CNF, DNF）
- Karnaugh Map
- Boolean Circuit